# WHEN IS THE AUTOMORPHISM GROUP OF AN AFFINE VARIETY LINEAR?

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Abstract. Let $\mathrm{Aut}_{\mathrm{alg}}(X)$ be the subgroup of the group of regular automorphisms $\mathrm{Aut}(X)$ of an affine algebraic variety $X$ generated by all connected algebraic subgroups. We prove that if $\dim X \geq 2$ and if $\mathrm{Aut}_{\mathrm{alg}}(X)$ is "rich enough", $\mathrm{Aut}_{\mathrm{alg}}(X)$ is not linear, i.e., it cannot be embedded into $\mathrm{GL}_n(\mathbb{F})$, where $\mathbb{K}$ is an algebraically closed field of characteristic zero and $\mathbb{F}$ is a field. Moreover, $\mathrm{Aut}(X)$ is isomorphic to an algebraic group as an abstract group only if the connected component of $\mathrm{Aut}(X)$ is either the algebraic torus or a direct limit of commutative unipotent groups. Finally, we prove that for an uncountable $\mathbb{K}$ the group of birational transformations of $X$ cannot be isomorphic to the group of automorphisms of an affine variety if $X$ is endowed with a rational action of a positive-dimensional linear algebraic group.

## 1. Introduction

In this paper we work over algebraically closed field $\mathbb{K}$ of characteristic zero, and $X$ always denotes an irreducible affine variety. It is well-known that the automorphism group of an affine variety may be very large. For example, the automorphism group $\mathrm{Aut}(\mathbb{A}^2)$ of the affine plane $\mathbb{A}^2$ contains a free product of two polynomial rings in one variable. Consequently, $\mathrm{Aut}(\mathbb{A}^2)$ is infinite-dimensional and cannot be given a structure of an algebraic group. Moreover, it is shown in [3, Proposition 2.3] that $\mathrm{Aut}(\mathbb{A}^2)$ is not linear, i.e. $\mathrm{Aut}(\mathbb{A}^2)$ cannot be embedded into the general linear group $\mathrm{GL}_n(\mathbb{K})$ as an abstract group. The first main result of the present note is a generalization of this statement to a big family of affine varieties.

It is well-known (Proposition 2.3) that the automorphism group $\mathrm{Aut}(X)$ has a structure of an *ind-group* (see Section 2.2 for the definition) and if $\dim X \geq 2$, $\mathrm{Aut}(X)$ is infinite-dimensional unless $\mathrm{Aut}(X)$ is a countable extension of the algebraic torus. But even if the automorphism group $\mathrm{Aut}(X)$ is infinite-dimensional it may happen that $\mathrm{Aut}(X)$ embeds into $\mathrm{GL}_n(\mathbb{K})$. For exmaple, [10, Example 6.14] shows that there is an affine surface $S$ such that $\mathrm{Aut}(S)$ is isomorphic to the polynomial ring in one variable $\mathbb{K}[t]$ and as an abstract additive group $\mathrm{Aut}(S)$ is isomorphic to the additive group of the base field and hence embeds into $\mathrm{GL}_2(\mathbb{K})$. However, if $\mathrm{Aut}(X)$ is rich enough, $\mathrm{Aut}(X)$ cannot be embedded into $\mathrm{GL}_n(\mathbb{K})$. More precisely, we prove the following statement.

We denote the additive and multiplicative group of the field $\mathbb{K}$ by $\mathbb{G}_a$ and $\mathbb{G}_m$ respectively. For a given affine variety $X$ we denote by $\mathrm{Aut}_{\mathrm{alg}}(X)$ the subgroup of $\mathrm{Aut}(X)$ generated by all connected algebraic subgroups.

**Theorem 1.1.** *Assume $X$ is at least two-dimensional variety such that $\mathrm{Aut}(X)$ contains an algebraic subtorus $T \simeq \mathbb{G}_m^k$, $k \geq 1$, a root subgroup $U \subset \mathrm{Aut}(X)$ and the invariant subrings $\mathcal{O}(X)^T, \mathcal{O}(X)^U \subset \mathcal{O}(X)$ do not coincide. Then $\mathrm{Aut}_{\mathrm{alg}}(X)$ cannot be embedded into $\mathrm{GL}_n(\mathbb{F})$ for any field $\mathbb{F}$.*

The assumption in Theorem 1.1 is necessary. Indeed, consider $X$ isomorphic to $\mathbb{A}^1 \times C$, where $C$ is a smooth affine curve having trivial automorphism group and no non-constant invertible regular functions. By Remark 3.2 $\mathrm{Aut}(X)$ is isomorphic to $\mathbb{G}_m \ltimes \mathcal{O}(C)^+$ which can be embedded into $\mathbb{K}(C)^* \ltimes \mathbb{K}(C)^+$, where $\mathbb{K}(C)$ is the function field of $C$.

If $X$ admits no $\mathbb{G}_m$-action, but admits two non-commuting $\mathbb{G}_a$-actions, $\mathrm{Aut}(X)$ can be embedded into $\mathrm{GL}_n(\mathbb{K})$. For example, there exists an affine surface $X$ (see [1, Example 4.1.3]) that has an automorphism group $\mathrm{Aut}(X) = \mathrm{Aut}_{\mathrm{alg}}(X) \simeq \mathbb{K}[x] * \mathbb{K}[y]$ which is linear by [11, Theorem] as additive groups $\mathbb{K}[x] \simeq \mathbb{K}[y]$ are isomorphic as abstract groups to $\mathbb{G}_a$.

The second question we study is whether the automorphism group of an affine variety can be isomorphic to a linear algebraic group. More precisely, we have the following statement which is the main result of the paper.

**Theorem 1.2.** *Let $X$ be an affine variety. If $\mathrm{Aut}(X)$ is isomorphic to a linear algebraic group as an abstract group, then the connected component $\mathrm{Aut}^\circ(X)$ is commutative. Moreover, in this case $\mathrm{Aut}^\circ(X)$ is either the algebraic torus or a direct limit of commutative unipotent groups.*

We denote by $\mathrm{Bir}(X)$ the group of birational transformations of $X$. It is well-known that such a group may be very large. For example the Cremona group $\mathrm{Bir}(\mathbb{A}^n) = \mathrm{Bir}(\mathbb{P}^n)$ for $n > 1$ is known to be very big, in particular, much larger than $\mathrm{Aut}(\mathbb{A}^n)$. Proposition 5.1 shows that the Cremona group $\mathrm{Bir}(\mathbb{A}^n) = \mathrm{Bir}(\mathbb{P}^n)$, $n > 0$, is not isomorphic to the automorphism group of any affine variety. Moreover, if $\mathrm{Bir}(X)$ is "rich enough", $\mathrm{Bir}(X)$ is also not isomorphic to the automorphism group of any affine variety. More precisely, we prove the following statement.

**Theorem 1.3.** *Assume $\mathbb{K}$ is uncountable and $X, Y$ are affine irreducible algebraic varieties. Assume $X$ is endowed with a rational action of a positive-dimensional linear algebraic group. Then the group of birational transformations $\mathrm{Bir}(X)$ is not isomorphic to $\mathrm{Aut}(Y)$.*

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## 2. Preliminaries

2.1. **Derivations and group actions.** Recall that $X$ is an irreducible affine algebraic variety. A derivation $\delta$ is called *locally finite* if it acts locally finitely on $\mathcal{O}(X)$, i.e., for any function $f \in \mathcal{O}(X)$ there is a finite-dimensional vector subspace $W \subset \mathcal{O}(X)$ such that $f \in W$ and $W$ is stable under action of $\delta$. A derivation $\delta \in \mathrm{Der}(\mathcal{O}(X))$ is called *locally nilpotent* if for any function $f \in \mathcal{O}(X)$ there exists $k \in \mathbb{N}$ (which depends on $f$) such that $\delta^k(f) = 0$. Note that there is a one-to-one correspondence between locally nilpotent derivations on $\mathcal{O}(X)$ and $\mathbb{G}_a$-actions on $X$ given by the map $\delta \mapsto \{t \mapsto \exp(t\delta)\}$.

An element $u \in \mathrm{Aut}(X)$ is called *unipotent* if $u = \exp(\partial)$ for some locally nilpotent derivation $\partial$.

2.2. **Ind-groups.** The notion of an ind-group goes back to Shafarevich who called it an infinite dimensional algebraic group (see [17]). We refer to [5] for basic notions in this context.

**Definition 2.1.** By an affine *ind-variety* we understand an injective limit $V = \varinjlim V_i$ of an ascending sequence $V_0 \hookrightarrow V_1 \hookrightarrow V_2 \hookrightarrow \ldots$ such that the following holds:

(1) $V = \bigcup_{k \in \mathbb{N}} V_k$;
(2) each $V_k$ is an affine algebraic variety;
(3) for all $k \in \mathbb{N}$ the embedding $V_k \hookrightarrow V_{k+1}$ is closed in the Zariski topology.

For simplicity we will call an affine ind-variety simply an ind-variety.

An ind-variety $V$ has a natural *topology*: a subset $S \subset V$ is called closed, resp. open, if $S_k := S \cap V_k \subset V_k$ is closed, resp. open, for all $k \in \mathbb{N}$. A closed subset $S \subset V$ has a natural structure of an ind-variety and is called an ind-subvariety.

A set theoretical product of ind-varieties admits a natural structure of an ind-variety. A *morphism* between ind-varieties $V = \bigcup_m V_m$ and $W = \bigcup_l W_l$ is a map $\phi : V \to W$ such that for every $m \in \mathbb{N}$ there is an $l \in \mathbb{N}$ such that $\phi(V_m) \subset W_l$ and that the induced map $V_m \to W_l$ is a morphism of algebraic varieties. This allows us to give the following definition.

**Definition 2.2.** An ind-variety $H$ is said to be an *ind-group* if the underlying set $H$ is a group such that the map $H \times H \to H$, defined by $(g, h) \mapsto gh^{-1}$, is a morphism of ind-varieties.

A *closed subgroup* $G$ of $H$ is a subgroup that is at the same time a closed subset. In this case $G$ is again an ind-group with respect to the induced ind-variety structure. A closed subgroup $G$ of an ind-group $H = \varinjlim H_i$ is called an *algebraic subgroup* if $G$ is contained in $H_i$ for some $i$.

The next result can be found in [5, Section 5].

**Proposition 2.3.** *Let $X$ be an affine variety. Then* $\mathrm{Aut}(X)$ *has the structure of an ind-group such that a regular action of an algebraic group $H$ on $X$ induces an ind-group homomorphism $H \to \mathrm{Aut}(X)$.*

2.3. **Root subgroups.** In this section we describe *root subgroups* of $\mathrm{Aut}(X)$ for a given affine variety $X$ with respect to a subtorus.

**Definition 2.4.** Let $T$ be a subtorus in $\mathrm{Aut}(X)$, i.e. a closed algebraic subgroup isomorphic to a torus. A closed algebraic subgroup $U \subset \mathrm{Aut}(X)$ isomorphic to $\mathbb{G}_a$ is called a *root subgroup* with respect to $T$ if the normalizer of $U$ in $\mathrm{Aut}(X)$ contains $T$.

Since $\mathbb{G}_a$ contains no non-trivial closed normal subgroups, every non-trivial regular action is faithful. Hence, such an algebraic subgroup $U$ corresponds a non-trivial *normalized* $\mathbb{G}_a$-action on $X$, i.e. a $\mathbb{G}_a$-action on $X$ whose image in $\mathrm{Aut}(X)$ is normalized by $T$.

Assume $U \subset \mathrm{Aut}(X)$ is a root subgroup with respect to $T$. Since $T$ normalizes $U$, we can define an action $\varphi\colon T \to \mathrm{Aut}(U)$ of $T$ on $U$ given by $t.u = t \circ u \circ t^{-1}$ for all $t \in T$ and $u \in U$. Moreover, since $\mathrm{Aut}(U) \simeq \mathbb{G}_m$, such an action corresponds to a character of the torus $\chi\colon T \to \mathbb{G}_m$, which does not depend on the choice of isomorphism between $\mathrm{Aut}(U)$ and $\mathbb{G}_m$. This character is called the *weight* of $U$. The algebraic subgroups $T$ and $U$ generate an algebraic subgroup in $\mathrm{Aut}(X)$ isomorphic to $\mathbb{G}_a \rtimes_\chi T$.

Consider a nontrivial algebraic action of $\mathbb{G}_a$ on $X$, given by $\lambda\colon \mathbb{G}_a \to \mathrm{Aut}(X)$. If $f \in \mathcal{O}(X)$ is $\mathbb{G}_a$-invariant, then the *modification* $f \cdot \lambda$ of $\lambda$ is defined in the following way:

$$(f \cdot \lambda)(r)x = \lambda(f(x)r)x$$

for $r \in \mathbb{C}$ and $x \in X$. This is again a $\mathbb{G}_a$-action. It is not difficult to see that if $X$ is irreducible and $f \neq 0$, then $f \cdot \lambda$ and $\lambda$ have the same invariants. If $U \subset \mathrm{Aut}(X)$ is a closed algebraic subgroup isomorphic to $\mathbb{G}_a$ and if $f \in \mathcal{O}(X)^U$ is a $U$-invariant, then in a similar way we define the modification $f \cdot U$ of $U$. Pick an isomorphism $\lambda\colon \mathbb{G}_a \to U$ and set

$$f \cdot U = \{(f \cdot \lambda)(r) \mid r \in \mathbb{G}_a\}.$$

2.4. **Divisible elements.** We call an element $f$ in a group $G$ *divisible by* $n$ if there exists an element $g \in G$ such that $g^n = f$. An element is called *divisible* if it is divisible by all $n \in \mathbb{Z}^+$. If $G$ is an agebraic group, then by [10, Lemma 3.12] for any $f \in G$ there exist $k > 0$ that depends on $f$ such that $f^k$ is a divisible element.

## 3. Proof of Theorem 1.1

The following lemma is well known and appeared in similar form in [4, Lemma 3.1].

**Lemma 3.1.** *Assume that $\mathfrak{g}$ is $\mathbb{Z}^r$-graded for $r > 0$ and consider a locally finite element $z \in \mathfrak{g}$ that does not belong to the zero component $\mathfrak{g}_0$. Then there exists a locally nilpotent homogeneous component of $z$ of non-zero weight.*

*Proof.* Let us take the convex hull $P(z) \subset \mathbb{Z}^r \otimes \mathbb{Q}$ of component weights of $z$. Then for any non-zero vertex $v \in P(z)$ the corresponding homogeneous component is locally nilpotent. The details are left to the reader. $\square$

*Proof of Theorem 1.1.* Since $U$ is a root subgroup with respect to $T$, $T$ acts on $U$ by conjugations which implies that $T$ acts on $\mathbb{O}(X)^U$. By assumption, there is a $T$-semi-invariant $f \in \mathcal{O}(X)^U$ of non-zero weight. Hence, $\{f^k \cdot U \subset \mathrm{Aut}(X) \mid k \in \mathbb{N}\}$ are root subgroups with respect to $T$ with different weights. Without loss of generality we can assume that $U$ is a root subgroup with respect to $T$ of non-zero weight since otherwise we can just replace $U$ by $f \cdot U$.

**Claim 1.** *The subgroup*

$$G = T \ltimes (\bigoplus_{k\geq 1} f^k \cdot U) \subset \mathrm{Aut}(X)$$

*is not linear.*

Indeed, assume towards a contradiction that the subgroup $G \subset \mathrm{Aut}(X)$ is linear, i.e., there is an embedding $\varphi\colon G \to \mathrm{GL}_n(\mathbb{K})$. Since $G$ is solvable, its image $\varphi(G) \subset \mathrm{GL}_n(\mathbb{K})$ is also solvable which implies that the closure $\overline{\varphi(G)} \subset \mathrm{GL}_n(\mathbb{K})$ is solvable too. Note that $\overline{\varphi(G)}$ is an algebraic subgroup of $\mathrm{GL}_n(\mathbb{K})$. Hence, the connected component $\overline{\varphi(G)}^{\circ}$ is conjugate to the Borel subgroup $B \subset \mathrm{GL}_n(\mathbb{K})$ of upper triangular matrices. Therefore, without loss of generality we can assume that $\overline{\varphi(G)}^{\circ} \subset B$. We claim that $\varphi(G) \subset \overline{\varphi(G)}^{\circ} \subset B$. Indeed, each element $g \in G$ belongs to an algebraic subgroup of $G$ and hence is divisible. Consequently, $\varphi(g) \in \overline{\varphi(G)}$ is divisible too. If $\varphi(g) \notin B$, $\varphi(g)$ belongs to a finite extension of $\varphi(B)$, i.e., can be written as a product $hb$, where $b \in B$ and $h$ is a non-trivial element of finite order. The product $hb$ can be divisible in $\varphi(G)$ if and only if $h$ is the identity element. We conclude that $\varphi(G) \subset B$ which porves the claim. Therefore, the commutator $[G, G] = \bigoplus_{k\geq 1} f^k \cdot U$ embedds into $[B, B]$. In other words $\varphi(\bigoplus_{k\geq 1} f^k \cdot U)$ is a subgroup of the unipotent radical of $B$.

Consider the closed subgroup $\overline{\varphi(T)}^{\circ} \ltimes \overline{\varphi(f^k \cdot U)} \subset B \subset \mathrm{GL}_n(\mathbb{K})$. The subgroup $\overline{\varphi(f^k \cdot U)} = \overline{\varphi(f^k \cdot U)}^{\circ} \subset [B, B]$ is unipotent and $\overline{\varphi(T)} \subset B$ is an algebraic subgroup. Hence, $\overline{\varphi(T)}^{\circ} \subset \overline{\varphi(T)}$ is a finite index subgroup which implies that $\overline{\varphi(T)}^{\circ}$ contains infinitely many elements of finite order of $\varphi(T)$. As a consequence, $\overline{\varphi(T)}^{\circ}$ contains a copy of algebraic subtorus of positive dimension. Pick a big enough $k \in \mathbb{N}$ such that the kernel of $T$-action on $f^k \cdot U$ is $\langle \xi_k \rangle$, where $\xi_k$ is an element of order bigger than the index $s = [\overline{\varphi(T)} : \overline{\varphi(T)}^{\circ}]$ and $\xi_k$ acts on $\mathbb{K}f$ non-trivially. Hence, $\xi_k^s \in \overline{\varphi(T)}^{\circ}$ and since $k$ is chosen to be big enough, we have that

$$\xi_k^s \text{ acts on } \mathbb{K}f \text{ non-trivially.} \tag{1}$$

Since $\varphi(\xi_k^s)$ centralizes $\varphi(f^k \cdot U)$, $\varphi(\xi_k^s)$ centralizes $\overline{\varphi(f^k \cdot U)}$ too. Choose a subtorus of $\overline{\varphi(T)}^\circ$ which we denote by $\tilde{T}$ that contains $\varphi(\xi_k^s)$. Pick $u_k \in \varphi(f^k \cdot U)$ and consider the unipotent subgroup $V_k = \langle \tilde{T}.u_k \rangle = \langle t u_k t^{-1} \mid t \in \tilde{T} \rangle \subset \mathrm{GL}_n(\mathbb{K})$. Note that $\tilde{T}$ normalizes $V_k$. Hence, the unipotet group $V_k$ is a direct product of root subgroups with respect to $\tilde{T}$. The kernel of $\tilde{T}$-action on $V_k$ contains $\langle \varphi(\xi_k^s) \rangle$. Since $\mathrm{GL}_n(\mathbb{K})$ is an algebraic group, i.e., is finitely dimensional, for a big enough $k$, the weights of all root subgroups of $V_k$ with respect to $\tilde{T}$ are the same as the weights of the root subgroups of $V_{k+1} = \langle \tilde{T}.u_{k+1} \rangle$, where $u_{k+1} \in \varphi(f^{k+1} \cdot U) \subset [B, B] \subset \mathrm{GL}_n(\mathbb{K})$. Hence, $\langle \varphi(\xi_k^s) \rangle$ acts trivially on $V_{k+1}$. As a consequence, $\langle \xi_k^s \rangle$ acts trivially on $\varphi^{-1}(V_k) \subset f^k \cdot U$ and on $\varphi^{-1}(V_{k+1}) \subset f^{k+1} \cdot U$. Therefore, $\langle \xi_k^s \rangle$ acts trivially on $\overline{\varphi^{-1}(V_k)} \subset f^k \cdot U$ and on $\overline{\varphi^{-1}(V_{k+1})} \subset f^{k+1} \cdot U$ which implies that $\langle \xi_k^s \rangle$ acts trivially on $\mathbb{K}f$. This contradicts (1) which proves the theorem if $X$ admits $\mathbb{G}_m$- and $\mathbb{G}_a$-actions.

If $X$ admits two non-commuting $\mathbb{G}_m$-actions, then by Lemma 3.1 $X$ admits a $\mathbb{G}_a$-action and the claim of the theorem follows from above. □

**Remark 3.2.** Let $C$ be a smooth affine curve having trivial automorphism group and no non-constant invertible regular functions. Then

$$\mathrm{Aut}(\mathbb{A}^1 \times C) = T \ltimes \mathcal{O}(C) \cdot U,$$

where $T = \{(x, y) \mapsto (ax, y) \mid a \in \mathbb{K}^*\}$ and $U = \{(x, y) \mapsto (x + b, y) \mid b \in \mathbb{K}\}$. Indeed, let $\varphi: \mathbb{A}^1 \times C \to \mathbb{A}^1 \times C$ be an automorphism of $\mathbb{A}^1 \times C$. By [10, Lemma 6.13] the second projection $\mathrm{pr}_2: \mathbb{A}^1 \times C \to C$ is invariant under automorphisms of $\mathbb{A}^1 \times C$. Hence, $\varphi(x, y) = (\psi(x, y), y)$ for all $x \in \mathbb{A}^1, y \in C$ and some morphism $\psi: \mathbb{A}^1 \times C \to \mathbb{A}^1$. For every $y \in C$, $\psi(\cdot, y): \mathbb{A}^1 \to \mathbb{A}^1$ is an isomorphism. Hence, $\psi(x, y) = a(y)x + b(y)$, where $a, b \in \mathcal{O}(C)$. Since $\varphi$ is an isomorphism, $a$ is an invertible regular function, i.e., $a \in \mathbb{K}^*$.

**Remark 3.3.** As it is already mentioned in the introduction, it is proved in [3] that $\mathrm{Aut}(\mathbb{A}^2)$ is not linear, i.e., it cannot be embedded into $\mathrm{GL}_n(\mathbb{K})$ for any $n \in \mathbb{N}$. This also follows from Theorem 1.1. Moreover, in [3, Proposition 2.3] it is proved that there is a countably generated subgroup of the subgroup

$$\mathrm{J} = \{(ax + c, by + f(x)) | a, b \in \mathbb{C}^*, c \in \mathbb{C}, f(y) \in \mathbb{C}[x]\} \subset \mathrm{Aut}(\mathbb{A}^2)$$

that is not linear. We note that by the Jung-Van der Kulk Theorem (see [8] and [9]) $\mathrm{Aut}(\mathbb{A}^2)$ is the amalgamated product of J and the group of affine transformations $\mathrm{Aff}_2$ of $\mathbb{A}^2$ along their intersection $C$, i.e.,

$$\mathrm{Aut}(\mathbb{A}^2) = \mathrm{Aff}_2 *_C \mathrm{J}\,. \tag{2}$$

Using such an amalgamated product we claim that any representation of a subgroup

$$\mathrm{SAut}(\mathbb{A}^2) = \left\{ (f, g) \in \mathrm{Aut}(\mathbb{A}^2) \mid \mathrm{jac}(f) = \det \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = 1 \right\}$$

is trivial, i.e., any homomorphism $\varphi: \mathrm{SAut}(\mathbb{A}^2) \to \mathrm{GL}_n(\mathbb{K})$ is trivial. To show this we first note that the amalgamated product structure of $\mathrm{Aut}(\mathbb{A}^2)$ induces the amalgamated product structure of $\mathrm{SAut}(\mathbb{A}^2)$. More precisely, $\mathrm{SAut}(\mathbb{A}^2)$ is the amalgamated product of the group $\mathrm{SAff}_2$ of special affine transformations of $\mathbb{A}^2$ and the subgroup

$$\mathrm{J}^s = \{(ax + c, by + f(x)) | a, b \in \mathbb{C}^*, ab = 1, c \in \mathbb{C}, f(y) \in \mathbb{C}[x]\} \subset \mathrm{Aut}(\mathbb{A}^2).$$

By [3, Proposition 2.3] there is no embedding of $\mathrm{J}^s$ into $\mathrm{GL}_n(\mathbb{K})$. Hence, there is a non-identity element $g \in \mathrm{J}^s$ such that $g$ is the kernel of $\varphi$. Therefore, the normal subgroup that contains $g$ is also in the kernel of $\varphi$. But by [6] any normal subgroup that contains $g$ coincides with $\mathrm{SAut}(\mathbb{A}^2)$ which proves the claim.

Moreover, any group homomorphism $\psi\colon \mathrm{Aut}(\mathbb{A}^2) \to \mathrm{GL}_n(\mathbb{K})$ factors through the homomorphism $\mathrm{jac}\colon \mathrm{Aut}(\mathbb{A}^2) \to \mathbb{G}_m$. Indeed, similarly as above, there is $g \in \mathrm{J}^s$ that is in the kernel of $\psi$. Hence, by the same argument as above the normal subgroup generated by $g$ contains $\mathrm{SAut}(\mathbb{A}^2)$ and the claim follows.

## 4. Automorphism group that is isomorphic to a linear algebraic group

We begin this section with the lemma that is used in the proof of Theorem 1.2.

**Lemma 4.1.** *Let $\tilde{U}, \tilde{V} \subset \mathrm{Aut}(\mathbb{A}^2)$ be two one-dimensional unipotent subgroups that act on $\mathbb{A}^2$ with different generic orbits. Then*

*(1) the subgroup $G_{\tilde{U}} \subset \mathrm{Aut}(\mathbb{A}^2)$ generated by all one-dimensional unipotent subgroups that have the same generic orbits as $\tilde{U}$ coincides with its centralizer;*

*(2) the subgroup generated by $G_{\tilde{U}}$ and $G_{\tilde{V}}$ cannot be presented as a finite product of $G_{\tilde{U}}$ and $G_{\tilde{V}}$.*

*Proof.* Recall that the group $\mathrm{Aut}(\mathbb{A}^2)$ has the amalgamated product structure $\mathrm{Aff}_2 *_C \mathrm{J}$, see (2). By [16] any closed algebraic subgroup of $\mathrm{Aut}(\mathbb{A}^2)$ is conjugate to one of the factors $\mathrm{Aff}_2$ or J. Since $G_{\tilde{U}} \subset \mathrm{Aut}(\mathbb{A}^2)$ is infinite-dimensional, $G_{\tilde{U}}$ is conjugate to a subgroup of J. Moreover, since $G_{\tilde{U}}$ is infinite-dimensional, commutative and consists of unipotent elements, $G_{\tilde{U}}$ is conjugate to a subgroup of

$$\mathrm{J}_u = \{(x, y + f(x)) \mid f(y) \in \mathbb{C}[x]\} \subset \mathrm{Aut}(\mathbb{A}^2).$$

Since $G_{\tilde{U}}$ is generated by all one-dimensional unipotent subgroups that have the same generic orbits, $G_{\tilde{U}}$ is conjugate to the whole $\mathrm{J}_u$. It is easy to check that $\mathrm{J}_u \subset \mathrm{Aut}(\mathbb{A}^2)$ coincides with its centralizer which proves (1).

Without loss of generality we can assume that $G_{\tilde{U}} = \mathrm{J}_u$. Since by (1) $G_{\tilde{V}}$ does not commute with $G_{\tilde{U}} = \mathrm{J}_u$, $G_{\tilde{V}}$ is not a subgroup of $\mathrm{J}_u$ and since $G_{\tilde{V}}$ is infinite-diemnsional, $G_{\tilde{V}}$ is not a subgroup of J. Hence, (2) follows from amalgamated product structure of $\mathrm{Aut}(\mathbb{A}^2)$. $\square$

Denote by $Q(R)$ the quotient field of a ring $R$. The next lemma is proved in [15, Lemma 1.1].

**Lemma 4.2.** *Let $U \subset \mathrm{Aut}(X)$ be a one-parameter unipotent subgroup. Then*

$$\mathrm{Cent}_{\mathrm{Aut}(X)}(\mathcal{O}(X)^U \cdot U) \subset \{f \cdot u \in \mathrm{Aut}(X) \mid f \in Q(\mathcal{O}(X)^U), u \in U\}.$$

*Proof of Theorem 1.2.* Assume $\varphi\colon G \to \mathrm{Aut}(X)$ is an isomorphism of abstract groups, where $G$ is a linear algebraic group. If the connected component $G^\circ \subset G$ is commutative, the finite index subgroup of $\mathrm{Aut}(X)$ is commutative. This implies that the connected component $\mathrm{Aut}^\circ(X)$ is commutative. Hence, by [2, Theorem B] (see also [14, Corollary 3.2]) $\mathrm{Aut}(X)^\circ$ is a union of commutative algebraic groups. The group $\mathrm{Aut}(X)^\circ$ either does not contain unipotent subgroups and in this case $\mathrm{Aut}^\circ(X)$ is isomorphic to an algebraic torus or $\mathrm{Aut}(X)^\circ$ contains unipotent subgroups. In the later case either $\mathrm{Aut}(X)$ does not contain a copy of $\mathbb{G}_m$ which implies that $\mathrm{Aut}^\circ(X)$ is the union of unipotent algebraic subgroups or $\mathrm{Aut}(X)$ contains a subgroup $T \times U$, where $T \simeq \mathbb{G}_m$ and $U \simeq \mathbb{G}_a$. We claim that in the second case $\mathrm{Aut}^\circ(X)$ is linear. Indeed, $U$ is a root subgroup with respect to $T$. Moreover, $\mathcal{O}(X)^T \neq \mathcal{O}(X)^U$. To prove this we first note that there is an open subset $S \subset X$ such that for any $x \in S$, $U.x \subset X$ is a one-dimensional orbit isomorphic to $\mathbb{A}^1$. Hence, if $T \times U.x \subset X$ is one-dimensional, $T \times U.x \simeq \mathbb{A}^1$ which is not possible as any coppies of $\mathbb{G}_m$ and $\mathbb{G}_a$ in the automorphism group of $\mathbb{A}^1$ do not commute. Therefore, $T \times U.x \subset X$ is two-dimensional. This implies that $\mathcal{O}(X)^U$ is a proper subalgebra of $\mathcal{O}(X)^{T\times U} = (\mathcal{O}(X)^U)^T$ as they have different Krull dimension by [5, Theorem 11.1.1.(7)]. We conclude that $\mathcal{O}(X)^U = \mathcal{O}(X)^T$ and by Theorem 1.1 $\mathrm{Aut}(X)$ is not linear which proves the theorem in case $G^\circ$ is commutative.

Assume now towards a contradiction that $G^\circ$ is non-commutative.

**Claim 2.** *$G$ contains closed connected commutative subgroups $U$ and $V$ that do not commute and $U$ normalizes $V$.*

Indeed, if $G$ is non-unipotent, it contains a maximal subtorus $T \subset G$ and a root subgroup normalized but not centralized by $T$. If $G$ is unipotent, then $G$ is nilpotent, i.e.,

$$G = G_0 \rhd G_1 \rhd \cdots \rhd G_n = \{\mathrm{id}\}, \tag{3}$$

where $G_{i+1} = [G, G_i]$, $[G, G_i] = \{ghg^{-1}h^{-1} \mid g \in G, h \in G_i\}$. In particular, $G_{n-1}$ is a subgroup of the center of $G$. Moreover, for any $H \subset G_{n-2} \setminus G_{n-1}$ isomorphic to $\mathbb{G}_a$, the group $V = H \times G_{n-1}$ is commutative. Choose a subgroup $U \subset G \setminus V$ isomorphic to $\mathbb{G}_a$ that does not commute with $V$. Note that such $U$ exists as $G$ is non-commutative. Moreover, we claim that $U$ normalizes $V$. Indded, $[U, V] \subset [G, G_{n-2}] = G_{n-1}$ which means that $uvu^{-1}v^{-1} \in G_{n-1}$ for any $u \in U$, $v \in V$. Hence, $uvu^{-1} \in G_{n-1}v \subset V$ which proves the claim.

Hence, $\overline{\varphi(U)}^\circ, \overline{\varphi(V)}^\circ \subset \mathrm{Aut}(X)$ are closed connected commutative subgroups. Since $\varphi(U)$ normalizes $\varphi(V)$, $\varphi(U)$ normalizes $\overline{\varphi(V)} \subset \mathrm{Aut}(X)$ and hence $\overline{\varphi(U)}^\circ$ normalizes $\overline{\varphi(V)}^\circ$. By [2, Theorem B] (see also [14, Corollary 3.2]) $\overline{\varphi(U)}^\circ, \overline{\varphi(V)}^\circ \subset \mathrm{Aut}(X)$ are unions of algebraic subgroups and hence $\overline{\varphi(U)}^\circ \ltimes \overline{\varphi(V)}^\circ$ is a union of algebraic subgroups. Note that $\overline{\varphi(U)}^\circ$ and $\overline{\varphi(V)}^\circ$ do not commute since otherwise the subgroups $\varphi^{-1}(\overline{\varphi(U)}^\circ)$ and $\varphi^{-1}(\overline{\varphi(V)}^\circ)$ of $G$ would commute which is not the case as $\varphi^{-1}(\overline{\varphi(U)}^\circ) \cap U \subset U$ is a dense subgroup and analogously $\varphi^{-1}(\overline{\varphi(U)}^\circ) \cap U \subset U$ is a dense subgroup. Therefore, there are non-commuting algebraic subgroups in $\mathrm{Aut}(X)$.

We have two possibilities, $\mathrm{Aut}(X)$ does not contain a copy of $\mathbb{G}_m$ or $\mathrm{Aut}(X)$ contains a copy of $\mathbb{G}_m$. Assume first that $\mathrm{Aut}(X)$ does not contain a copy of $\mathbb{G}_m$, then the semidirect product $\overline{\varphi(U)}^\circ \ltimes \overline{\varphi(V)}^\circ$ is the union of unipotent subgroups and in particular it contains a unipotent subgroup $W$ that acts on $X$ with a two-dimensional orbit $O$ that is isomorphic to $\mathbb{A}^2$, see [5, Theorem 11.1.1]. Pick subgroups $\tilde{U} \subset W$ and $\tilde{V} \subset W$ isomorphic to $\mathbb{G}_a$ that generate the algebraic subgroup that acts with a two-dimensional orbit $O \simeq \mathbb{A}^2$. Take the maximal commutative subgroups $H_1$ and $H_2$ of $\mathrm{Aut}(X)$ that contain $\mathcal{O}(X)^{\tilde{U}} \cdot \tilde{U}$ and $\mathcal{O}(X)^{\tilde{V}} \cdot \tilde{V} \subset \mathrm{Aut}(X)$ respectively. Therefore, $\varphi^{-1}(H_1), \varphi^{-1}(H_2) \subset G$ are closed subgroups. Hence, the subgroup $H \subset G$ generated by $\varphi^{-1}(H_1)$ and $\varphi^{-1}(H_2)$ can be presented as a finite product of subgroups $\varphi^{-1}(H_1)$ and $\varphi^{-1}(H_2)$. On the other hand, the subgroup of $\mathrm{Aut}(X)$ generated by $H_1$ and $H_2$ cannot be presented as a finite product of subgroups $H_1$ and $H_2$. Indeed, by Lemma 4.1(1) the restriction of $H_1$ to $O \simeq \mathbb{A}^2$ is the subgroup of $\mathrm{Aut}(O \simeq \mathbb{A}^2)$ generated by all one-dimensional unipotent subgroups that have the same generic orbits as $\tilde{U}\mid_O$. Analogous situation we have with $H_2$. Now by Lemma 4.1(2), the subgroup $H = \langle H_1, H_2\rangle \subset \mathrm{Aut}(X)$ restricted to $O \simeq \mathbb{A}^2$ cannot be presented as a finite product of $H_1$ and $H_2$ restricted to $O$. We arrive to the contradiction and finish the proof.

We are left with the case when $\mathrm{Aut}(X)$ contains a subgroup $T \simeq \mathbb{G}_m$. Since $\mathrm{Aut}_{\mathrm{alg}}(X)$ is not commutative, by [13, Theorem 1.3] there exists a copy of $\mathbb{G}_a$ in $\mathrm{Aut}(X)$. Now, by Lemma 3.1 there is root subgroup $U \subset \mathrm{Aut}(X)$ with respect to $T$. By Theorem 1.1 we can assume that $\mathcal{O}(X)^T = \mathcal{O}(X)^U$. Moreover, the maximal commutative subgroup that contains $T$ does not contain any algebraic subgroup different from $T$. Now, $\varphi^{-1}(T) \subset G$ acts on $\varphi^{-1}(\mathrm{Cent}_{\mathrm{Aut}(X)}(\mathcal{O}(X)^U \cdot U)) \subset G$ by conjugations. By Lemma 5 $\mathrm{Cent}_{\mathrm{Aut}(X)}(\mathcal{O}(X)^U \cdot U)$ coincides with its centalizer. Hence, $\varphi^{-1}(\mathrm{Cent}_{\mathrm{Aut}(X)}(\mathcal{O}(X)^U \cdot U)) \subset G$ also coincides with its centralizer and does not contain elements of finite order. Therefore, $\varphi^{-1}(\mathrm{Cent}_{\mathrm{Aut}(X)}(\mathcal{O}(X)^U \cdot U)) \subset G$ is a closed unipotent subgroup. Moreover, $\overline{\varphi^{-1}(T)} \subset G$ is the closed commutative algebraic subgroup that contains infinitely many elements of finite order which implies that $\overline{\varphi^{-1}(T)}^{\circ} = D \times V$, where $D \subset G$ is a torus of positive diemnsion and $V \subset G$ is a unipotent commutative subgroup.

**Claim 3.** *The group $\varphi(D \times V)$ is a subgroup of $\varphi(T)$.*

Indeed, $D \times V \subset \overline{\varphi^{-1}(T)}$ is a finite index subgroup. Hence, $\varphi(D \times V) \cap T \subset T$ is a finite index subgroup. Moreover, each element of $D \times V$ is divisible as $D \times V$ is the algebraic group. This implies that each element of $\varphi(D \times V)$ is divisible in $\varphi(D \times V)$. Consequently, each element of the intersection $T \cap \varphi(D \times V)$ is divisible and so $T \cap \varphi(D \times V)$ is a finite index subgroup of $T$ that consists of divisible elements of $T$. Therefore, $T \cap \varphi(D \times V) = T$.

**Claim 4.** *The unipotent subgroup $V \subset \varphi^{-1}(\mathrm{Cent}_{\mathrm{Aut}(X)}(\mathcal{O}(X)^U \cdot U)) \subset G$ is trivial.*

Indeed, by Lie-Kolchin Theorem [7, §17.6] the unipotent group $V$ acts on a unipotent subgroup $\varphi^{-1}(\mathrm{Cent}_{\mathrm{Aut}(X)}(\mathcal{O}(X)^U \cdot U)) \subset G$ with a fixed point which is not possible as $\varphi(V) \subset T$, and $T$ acts with a finite kernel on any element of

$$\{f \cdot u \in \mathrm{Aut}(X) \mid f \in Q(\mathcal{O}(X)^U), u \in U\}.$$

This proves the claim.

The algebraic subtorus $D \subset G$ acts on a unipotent subgroup $\varphi^{-1}(\mathrm{Cent}_{\mathrm{Aut}(X)}(\mathcal{O}(X)^U \cdot U))$ and hence $\varphi^{-1}(\mathrm{Cent}_{\mathrm{Aut}(X)}(\mathcal{O}(X)^U \cdot U))$ decomposes into a direct sum of finitely many one-dimensional $D$-invariant unipotent subgroups $V_1, \ldots, V_k$, $k \geq 1$. The subgroups

$$\varphi(V_i) \subset \{f \cdot u \in \mathrm{Aut}(X) \mid f \in Q(\mathcal{O}(X)^U), u \in U\}$$

are invariant under $\varphi(D)$-action and in particular because $T \subset \varphi(D)$, $\varphi(V_i)$ are invariant under the action of $T$. Hence, $\varphi(V_i) = R_i \cdot U$, where $R_i \subset Q(\mathcal{O}(X)^U)$ is a vector subspace for each $i = 1, \ldots, k$. Therefore, at least one $R_i$ is infinite-dimensional vector subspace of $Q(\mathcal{O}(X)^U)$, say $R_1$. Since $D$ acts on $V_1 \setminus \{0\}$ transitively, $\varphi(D)$ acts on $\varphi(V_1) \setminus \{0\}$ transitively too. But since $R_1$ is a infinite-dimensional vector space, there is no commutative subgroup of $\mathrm{GL}(R_1)$ that acts transitively on $R_1 \setminus \{0\}$. We arrive to a contradiction which finishes the proof. □

## 5. Proof of Theorem 1.3

We start this section with the next proposition.

**Proposition 5.1.** *Assume $\mathbb{K}$ is uncountable and $X$ is a connected affine variety. Then $\mathrm{Aut}(X)$ is not isomorphic to the Cremona group $\mathrm{Bir}(\mathbb{A}^n) = \mathrm{Bir}(\mathbb{P}^n)$ as an abstract group for any $n > 0$.*

*Proof.* The proof of this statement is similar to the proof of Theorem A in [2]. We give some details here for the convenience of the reader. Let

$$\mathrm{Tr} = \{(x_1, \ldots, x_n) \mapsto (x_1 + c_1, \ldots, x_n + c_n) \mid c_i \in \mathbb{K}\} \subset \mathrm{Aut}(\mathbb{A}^n) \subset \mathrm{Bir}(\mathbb{A}^n)$$

be the subgroup of all translations and $\mathrm{Tr}_i$ be the subgroup of translations of the $i$-th coordinate:

$$(x_1, \dots, x_n) \mapsto (x_1, \dots, x_i + c, \dots, x_n), \tag{4}$$

where $c$ in $\mathbb{K}$. Let $\mathrm{T} \subset \mathrm{GL}_n(\mathbb{K}) \subset \mathrm{Aut}(\mathbb{A}^n) \subset \mathrm{Bir}(\mathbb{P}^n)$ be the diagonal group (viewed as a maximal torus) and let $\mathrm{T}_i$ be the subgroup of automorphisms

$$(x_1, \dots, x_n) \mapsto (x_1, \dots, ax_i, \dots, x_n), \tag{5}$$

where $a \in \mathbb{K}^*$. A direct computation shows that Tr *(resp. $T$) coincides with its centralizer in* $\mathrm{Bir}(\mathbb{A}^n) = \mathrm{Bir}(\mathbb{P}^n)$. Assume towards a contradiction that there is an isomorphism $\varphi\colon \mathrm{Bir}(\mathbb{A}^n) \to \mathrm{Aut}(X)$ of abstract groups. Similarly as in [2, Lemma 5.2] the groups $\varphi(\mathrm{Tr})$, $\varphi(\mathrm{Tr}_i)$, $\varphi(T)$ and $\varphi(T_i)$ are closed subgroups of $\mathrm{Aut}(X)$ for all $i = 1, \dots, n$. Now the proof of [2, Theorem A] implies that $X \simeq \mathbb{A}^n$ and $\varphi(T) \subset \mathrm{Aut}(X \simeq \mathbb{A}^n)$ is isomorphic to the $n$-dimensional algebraic torus. Assume $U \subset \mathrm{PGL}_{n+1}(\mathbb{K}) \subset \mathrm{Bir}(\mathbb{A}^n)$ is a root subgroup with respect to $T$. This means that $T$ acts on $U$ with two orbits. Therefore, $\varphi(U) \subset \mathrm{Aut}(X)$ is a constructible subset which is a group. We conclude that $\varphi(U) \subset \mathrm{Aut}(X)$ is an algebraic subgroup. Moreover, since $\mathrm{PGL}_{n+1}(\mathbb{K})$ is generated by its finitely many root subgroups $U$ with respect to $T$, $\varphi(\mathrm{PGL}_{n+1}(\mathbb{K}))$ is generated by finitely many algebraic subgroups $\varphi(U)$ which implies that $\varphi(\mathrm{PGL}_{n+1}(\mathbb{K})) \subset \mathrm{Aut}(X \simeq \mathbb{A}^n)$ is an algebraic subgroup. Further, $\varphi(T) \subset \varphi(\mathrm{PGL}_{n+1}(\mathbb{K}))$ is a maximal subtorus that is isomorphic to $\mathbb{G}_m^n$ which means that $\varphi(\mathrm{PGL}_{n+1}(\mathbb{K}))$ is a simple algebraic group of rank $n$ that is isomorphic to $\mathrm{PGL}_{n+1}(\mathbb{K})$ as an abstract group. We conclude that $\varphi(\mathrm{PGL}_{n+1}(\mathbb{K}))$ is isomorphic to $\mathrm{PGL}_{n+1}(\mathbb{K})$ as an algebraic group. But this is not possible since the algebraic group $\mathrm{PGL}_{n+1}(\mathbb{K})$ does not act regularly on $\mathbb{A}^n$ as the only closed subgroup $H$ of codimension $\leq n$ of $\mathrm{PGL}_{n+1}(\mathbb{K})$ is a maximal parabolic subgroup such that $\mathrm{PGL}_{n+1}(\mathbb{K})/H \simeq \mathbb{P}^n$. We arrive to a contradiction with the isomorphism $X \simeq \mathbb{A}^n$. The proof follows. □

*Proof of Theorem 1.3.* Let $H \subset \mathrm{Bir}(X)$ be a maximal algebraic subtorus. By [12, Theorem 1] $X$ is birationally equivalent to $\mathbb{A}^l \times Z$, where $H$ acts on $\mathbb{A}^l$ with an open orbit and $Z \subset \mathbb{A}^r$ is an affine variety with a trivial action of $H$. By Proposition 5.1 we can assume that $Z$ is positive dimensional. Assume there is an isomorphism $\varphi\colon \mathrm{Bir}(X) = \mathrm{Bir}(\mathbb{A}^l \times Z) \to \mathrm{Aut}(Y)$. Consider the maximal commutative subgroup $G$ of $\mathrm{Bir}(\mathbb{A}^l \times Z)$ of the form

$$\{(x_1, \dots, x_l, z_1, \dots, z_r) \mapsto (f_1(z)x_1, \dots, f_l(z)x_l, g_1(z), \dots, g_r(z)) \mid f_i(z), g_i(z) \in \mathbb{K}(Z)\}$$

that contains the commutative subgroup

$$\{(x_1, \dots, x_l, z_1, \dots, z_r) \mapsto (f_1(z)x_1, \dots, f_l(z)x_l, z_1, \dots, z_r) \mid f_i(z) \in \mathbb{K}(Z)\},$$

where the map

$$Z \to Z \ \ (z_1, \dots, z_r) \mapsto (g_1(z), \dots, g_r(z))$$

is a birational transformation of $Z$.

**Claim 5.** *The subgroup $G \subset \mathrm{Bir}(\mathbb{A}^l \times Z)$ coincides with its centralizer.*

To prove this claim consider a birational transformation $\phi$ of $\mathbb{A}^l \times Z$ of the form

$$(x_1, \dots, x_l, z_1, \dots, z_r) \mapsto (F_1(x, z), \dots, F_l(x, z), G_1(x, z), \dots, G_r(x, z)),$$

where $F_i(x, z), G_i(x, z) \in \mathbb{K}(\mathbb{A}^l \times Z)$, $x = (x_1, \dots, x_l)$, $z = (z_1, \dots, z_r)$ that commutes with each element from $G$. Hence, $\phi$ commutes with $T \subset G$, i.e., with all birational transformations of $\mathbb{A}^l \times Z$ of the form

$$(x_1, \dots, x_l, z_1, \dots, z_r) \mapsto (t_1x_1, \dots, t_lx_l, z_1, \dots, z_r), \ \ t_1, \dots, t_r \in \mathbb{K}^*.$$

Direct computations show that

$$t_iF_i(t_1^{-1}x_1,\dots,t_l^{-1}x_l,z_1,\dots,z_r)=F_i(x_1,\dots,x_l,z_1,\dots,z_r)$$

and

$$G_j(t_1^{-1}x_1,\dots,t_l^{-1}x_l,z_1,\dots,z_r)=G_j(x_1,\dots,x_l,z_1,\dots,z_r)$$

for all $t_1,\dots,t_r\in\mathbb{K}^*$, $i=1\dots,l$, $j=1,\dots,r$. Therefore, $F_i(x,z)=h_i(z)x_i$ for some $h_i(z)\in\mathbb{K}(Z)$ and $G_j\in\mathbb{K}(Z)$. This proves the claim.

By [10, Lemma 2.4] $\varphi(G)\subset\operatorname{Aut}(Y)$ is a closed ind-subgroup and by [2, Theorem B] (see also [14, Corollary 3.2]) the connected component $\varphi(G)^\circ\subset\operatorname{Aut}(Y)$ is the union of commutative algebraic subgroups. Since $\varphi(G)^\circ\subset\varphi(G)$ is a countable index subgroup, there is an element $g=(f_1(z)x_1,x_2,\dots,x_l,z_1,\dots,z_r)\in G$ with non-constant $f_1$ such that $\varphi(g)$ belongs to $\varphi(G)^\circ$. Since $\varphi(G)^\circ$ is the union of connected algebraic groups, $\varphi(g)$ belongs to a connected algebraic subgroup of $\varphi(G)^\circ$ and hence there exists $k>0$ such that $\varphi(g)^k$ is a divisible element (see Section 2.4). The element $g^k$ again has a form $(\tilde{f}_1(z)x_1,x_2,\dots,x_l,z_1,\dots,z_r)$ with a non-constant $\tilde{f}_1=\frac{r_1}{r_2}$, where $r_1,r_2\in\mathcal{O}(Z)$. Moreover, $g^k\in G$ is not divisible. More precisely, without loss of generality we can assume that $r_1$ is non-constant and hence, there is no $h\in G$ such that $h^{\deg r_1+1}=g^k$. Indeed, otherwise there would exist a rational function $s\in\mathbb{K}(Z)$ such that $s^{\deg r_1+1}=\tilde{f}_1=\frac{r_1}{r_2}$ which is not the case. We get the contradiction which proves the theorem. $\square$

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