BRACKET WIDTH OF THE LIE ALGEBRA OF VECTOR FIELDS ON A SMOOTH AFFINE CURVE

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Abstract. We prove that the bracket width of the simple Lie algebra of vector fields $\mathrm{Vec}(C)$ of a smooth irreducible affine curve C with a trivial tangent sheaf is at most three. In addition, if C is a plane curve, the bracket width of $\mathrm{Vec}(C)$ is at most two and if moreover C has a unique place at infinity, the bracket width of $\mathrm{Vec}(C)$ is exactly two. We also show that in case C is rational, the width of $\mathrm{Vec}(C)$ equals one.

1. Introduction

Given a Lie algebra L over an infinite field $\Bbbk$, we define its bracket width as the supremum of lengths $\ell(x)$, where x is runs over the derived algebra $[L, L]$ and $\ell(x)$ is defined as the smallest number n of Lie brackets $[y_i, z_i]$ needed to represent x in the form

$$\sum_{i=1}^{n} [y_i, z_i].$$

The bracket width applies in studying different aspects of Lie algebras, see [Rom16]. In particular, in [Rom16] the author provides many examples of Lie algebras with the bracket width strictly bigger than one. However, the first example of a simple Lie algebra with the bracket width strictly bigger than one was found only very recently in [DKR21, Theorem A] among Lie algebras of vector fields of smooth affine curves which are simple by [Jor86] and [Sie96, Proposition 1]. In the current note we provide an upper bound on the bracket width of a Lie algebra of vector fields on an irreducible smooth affine curve C with certain properties. Our first main result is the following statement which partially answers [DKR21, Question 2].

Theorem A. *Let C be an irreducible smooth affine curve with trivial tangent sheaf. Then the bracket width of the Lie algebra $\mathrm{Vec}(C)$ is smaller than or equal to three. In addition, if C is a plane curve, the bracket width of $\mathrm{Vec}(C)$ is smaller than or equal to two.*

The upper bound for the bracket width given in Theorem A is, in particular, of interest since it allows us to compute the bracket width of $\mathrm{Vec}(C)$ for a certain family of smooth plane affine curves.

Corollary 1. *Let C be an irreducible non-rational smooth plane affine curve with a unique place at infinity. Then the width of the simple Lie algebra $\mathrm{Vec}(C)$ equals two.*

There are many examples of affine curves with only one place at infinity (see Definition 1), and they were studied in many different contexts, see, e.g., a paper of Kollár [Kol20] and the references therein. A simple class of examples is given by affine hyperelliptic plane curves

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$C \subset \mathbb{A}^2$ defined by equations $y^2 = h(x)$, where $h(x)$ is a monic polynomial of odd degree strictly greater than one which has only simple roots ([DKR21, Example 2]).

We believe that the assumption on the curve C in Corollary 1 can be lightened and we have the following conjecture.

Conjecture 1. *Assume C is a non-rational affine smooth plane curve. Then the bracket width of* $\mathrm{Vec}(C)$ *is exactly two.*

Assume f is a regular function on C. We define a principal open subset $C_f \subset C$ as

$$\{x \in C \mid f(x) \neq 0\} \subset C.$$

Note that C_f is a smooth affine curve itself.

Theorem B. *Let C be an irreducible smooth affine curve and C_f be its principal open subset. Then the bracket width of* $\mathrm{Vec}(C_f)$ *is smaller than or equal to the bracket width of* $\mathrm{Vec}(C)$.

We do not know an example of a smooth affine curve C with a principal open subset $U \subset C$ such that the width of $\mathrm{Vec}(U)$ is strictly smaller than $\mathrm{Vec}(C)$.

As a consequence of Theorem B we have the following statement that disproves [DKR21, Conjecture 1].

Corollary 2. *If $\Bbbk$ is algebraically closed, the bracket width of the Lie algebra* $\mathrm{Vec}(C)$ *of a rational smooth affine curve C is one.*

2. Proof of Theorem A

Proof of Theorem A. Denote by $\mathcal{O}(C)$ the ring of regular functions on C. Since by hypothesis the tangent sheaf of C is trivial, we have $\mathrm{Vec}(C) = \mathcal{O}(C) \cdot \tau$ for a certain nowhere vanishing global vector field $\tau \in \mathrm{Vec}(C)$, unique up to multiplication by a nonzero constant. It is well-known that every smooth affine variety of dimension d can be embedded into $\mathbb{A}^{2d+1}$ and that the bound $2d+1$ is optimal ([Sr91, Corollary 1]). In particular, a smooth affine curve can be embedded into $\mathbb{A}^3$ and this bound is sharp. Hence, $\mathcal{O}(C) \simeq \Bbbk[x,y,z]/I$, where $I \subset \Bbbk[x,y,z]$ is some ideal and we have the natural surjections

$$\pi \colon \Bbbk[x,y,z] \twoheadrightarrow \Bbbk[x,y,z]/I$$

and

$$\pi_* \colon \left\{\nu \in \mathrm{Der}\,\Bbbk[x,y,z] = \mathrm{Vec}(\mathbb{A}^3) \mid \nu(I) \subset I\right\} \twoheadrightarrow \mathrm{Vec}(C).$$

Note that $\{\nu \in \mathrm{Der}\,\Bbbk[x,y,z] \mid \nu(I) \subset I\} \subset \mathrm{Der}\,\Bbbk[x,y,z]$ is a Lie subalgebra and π_* is a homomorphism of Lie algebras. Then τ is the image of a derivation $\tilde{\tau} = \tilde{P}\frac{\partial}{\partial x} + \tilde{Q}\frac{\partial}{\partial y} + \tilde{R}\frac{\partial}{\partial z}$ that preserves the ideal I. Further, define $P, Q, R \in \mathcal{O}(C)$, $P = \pi(\tilde{P})$, $Q = \pi(\tilde{Q})$, $R = \pi(\tilde{R})$.

Claim 1.

$$\left\{[\tilde{\tau}, \tilde{f}\tilde{\tau}] + [y\tilde{\tau}, \tilde{g}\tilde{\tau}] + [z\tilde{\tau}, \tilde{h}\tilde{\tau}] \mid \tilde{f}, \tilde{g}, \tilde{h} \in \Bbbk[x,y,z]\right\} = (\tilde{P}, \tilde{Q}, \tilde{R})\tau,$$

where $(\tilde{P}, \tilde{Q}, \tilde{R})$ denotes the ideal of $\Bbbk[x,y,z]$ generated by $\tilde{P}, \tilde{Q}$ and $\tilde{R}$.

Indeed,

$$[\tilde\tau,\tilde f\tilde\tau]+[y\tilde\tau,\tilde g\tilde\tau]+[z\tilde\tau,\tilde h\tilde\tau]=(\tilde P\tilde f'_x+\tilde Q\tilde f'_y+\tilde R\tilde f'_z+ \tag{1}$$
$$y(\tilde P\tilde g'_x+\tilde Q\tilde g'_y+\tilde R\tilde g'_z)-\ \tilde g\tilde Q+z(\tilde P\tilde h'_x+\tilde Q\tilde h'_y+\tilde R\tilde h'_z)-\tilde h\tilde R)\tilde\tau$$

which equals

$$(\tilde P(\tilde f'_x+y\tilde g'_x+z\tilde h'_x)+\tilde Q(\tilde f'_y+y\tilde g'_y+z\tilde h'_y-\tilde g)+\tilde R(\tilde f'_z+y\tilde g'_z+z\tilde h'_z-\tilde h))\tilde\tau \tag{2}$$
$$=(\tilde P(\tilde f+y\tilde g+z\tilde h)'_x+\tilde Q((\tilde f+y\tilde g+z\tilde h)'_y-2\tilde g)+\tilde R((\tilde f+y\tilde g+z\tilde h)'_z-2\tilde h))\tilde\tau.$$

Define $\tilde r=\tilde f+y\tilde g+z\tilde h\in\mathcal{O}(C)$. Now, the expression (2) can be written as

$$(\tilde P\tilde r'_x+\tilde Q(\tilde r'_y-2\tilde g)+\tilde R(\tilde r'_z-2\tilde h))\tilde\tau.$$

For any $F,G,H\in\Bbbk[x,y,z]$ there exist $\tilde r,\tilde g,\tilde h$ such that $F=\tilde r'_x$, $G=\tilde r'_y-2\tilde g$ and $H=\tilde r'_z-2\tilde h$. Hence,

$$\Big\{(\tilde P\tilde r'_x+\tilde Q(\tilde r'_y-2\tilde g)+\tilde R(\tilde r'_z-2\tilde h))\tilde\tau\mid\tilde r,\tilde g,\tilde h\in\Bbbk[x,y,z]\Big\}=$$
$$\Big\{(\tilde PF+\tilde QG+\tilde RH)\tilde\tau\mid F,G,H\in\Bbbk[x,y,z]\Big\}=(\tilde P,\tilde Q,\tilde R)\tilde\tau.$$

This proves Claim 1.

We claim now that any vector field from $\mathrm{Vec}(C)$ can be written as the sum

$$[\tau,f\tau]+[\pi(y)\tau,g\tau]+[\pi(z)\tau,h\tau] \tag{3}$$

for some regular functions $f,g,h\in\mathcal{O}(C)$. Indeed, $(\tilde P,\tilde Q,\tilde R)$ is preserved by $\tilde\tau$, hence $\pi((\tilde P,\tilde Q,\tilde R))$ is preserved by τ. Whence $\pi((\tilde P,\tilde Q,\tilde R))\tau$ is the ideal in $\mathrm{Vec}(C)$. Therefore, because of simplicity of $\mathrm{Vec}(C)$ we have

$$\pi_*((\tilde P,\tilde Q,\tilde R)\tau)=\mathrm{Vec}(C)\text{ or equivalently }\pi((\tilde P,\tilde Q,\tilde R))=\mathcal{O}(C).$$

Finally, by Claim 1 we have

$$\Big\{\pi_*([\tilde\tau,\tilde f\tilde\tau])+\pi_*([y\tilde\tau,\tilde g\tilde\tau])+\pi_*([z\tilde\tau,\tilde h\tilde\tau])\mid\tilde f,\tilde g,\tilde h\in\Bbbk[x,y,z]\Big\}=\pi_*((\tilde P,\tilde Q,\tilde R)\tau)=\mathrm{Vec}(C).$$

Thus, the first statement of the theorem follows as π_* is a homomorphism of Lie algebras.

If C is a plane curve, then, similarly as above, $\mathrm{Vec}(C)=\mathcal{O}(C)\cdot\tau$, $\mathcal{O}(C)\simeq\Bbbk[x,y]/I$, where $I\subset\Bbbk[x,y]$ is an ideal and we have the natural surjections

$$\pi\colon\Bbbk[x,y]\twoheadrightarrow\Bbbk[x,y]/I$$

and

$$\pi_*\colon\ \big\{\nu\in\mathrm{Der}\,\Bbbk[x,y]=\mathrm{Vec}(\mathbb{A}^2)\mid\nu(I)\subset I\big\}\twoheadrightarrow\mathrm{Vec}(C).$$

Then τ is the image of a derivation $\tilde\tau=\tilde P\frac{\partial}{\partial x}+\tilde Q\frac{\partial}{\partial y}$ that preserves the ideal I. Further, define $P,Q\in\mathcal{O}(C)$, $P=\pi(\tilde P)$, $Q=\pi(\tilde Q)$. The second statement of the theorem follows if we prove that any vector field of $\mathrm{Vec}(C)$ can be written as the sum

$$[\tau,f\tau]+[\pi(y)\tau,g\tau] \tag{4}$$

for some regular functions $f,g\in\mathcal{O}(C)$. Similarly as above (4) follows from the next equality

$$\Big\{[\tilde\tau,\tilde f\tilde\tau]+[y\tilde\tau,\tilde g\tilde\tau]\mid\tilde f,\tilde g,\in\Bbbk[x,y]\Big\}=(\tilde P,\tilde Q)\tilde\tau$$

which is proved analogously as Claim 1. ∎

Definition 1. An irreducible smooth affine curve C is said to have *a unique place at infinity* if it is equal to the complement of a single closed point in a smooth projective curve $\bar{C}$.

Proof of Corollary 1. By [DKR21, Theorem A] the bracket width of $\mathrm{Vec}(C)$ is strictly greater than one. Since C is a smooth irreducible plane affine curve, its tangent sheaf is trivial. Now by Theorem A the width of $\mathrm{Vec}(C)$ is less or equal than two which proves the claim. □

3. Proof of Theorem B

Proof of Theorem B. Denote by $\mathcal{K}(C)$ the field of fractions of $\mathcal{O}(C)$. Then

$$\mathcal{K}(C) \otimes_{\mathcal{O}(C)} \mathrm{Der}(\mathcal{O}(C)) \simeq \mathrm{Der}(\mathcal{K}(C)) \tag{5}$$

and the latter is known to be a free $\mathcal{K}(C)$-module of rank equal to $\dim C = 1$. By (5) $\mathrm{Vec}(C) = \mathrm{Der}\,\mathcal{O}(C) \subset \mathrm{Der}\,\mathcal{K}(C) = \mathcal{K}(C)\tau$, where $\tau \in \mathrm{Vec}(C)$ is a global vector field. Hence, $\mathrm{Vec}(C) = \mathcal{R}(C) \cdot \tau$ for a certain $\mathcal{O}(C)$-module $\mathcal{R}(C) \subset \mathcal{K}(C)$. As a consequence, $\mathrm{Vec}(C_f) = \mathcal{R}(C)[f^{-1}]\tau$. Now, for $a, b \in \mathcal{R}(C)$, $\frac{a}{f^k}\tau, \frac{b}{f^k}\tau \in \mathrm{Vec}(C_f) = \mathcal{R}(C)[f^{-1}]\tau$ and we have:

$$\left[\frac{a}{f^k}\tau, \frac{b}{f^k}\tau\right] = \left(\frac{a}{f^k}\tau(\frac{b}{f^k}) - \frac{b}{f^k}\tau(\frac{a}{f^k})\right)\tau = \frac{1}{f^{2k}}\,[a,b]\,\tau. \tag{6}$$

Further, for any $h \in \mathcal{R}(C)[f^{-1}]$ there exists $g \in \mathcal{R}(C)$ such that

$$h = \frac{g}{f^{2k}}$$

for some $k \in \mathbb{N}$. Assume that

$$g\tau = [a_1\tau, b_1\tau] + \cdots + [a_n\tau, b_n\tau].$$

Then using (6) we have

$$h\tau = \frac{g}{f^{2k}}\tau = \left[\frac{a_1}{f^k}\tau, \frac{b_1}{f^k}\tau\right] + \cdots + \left[\frac{a_n}{f^k}\tau, \frac{b_n}{f^k}\tau\right].$$

This completes the proof. □

Proof of Corollary 2. Let us recall that every rational smooth affine curve C is isomorphic to $\mathbb{A}^1 \setminus \Lambda$, where Λ is a finite set of $r \geq 0$ points. In particular, it admits a closed embedding into $\mathbb{A}^2$. Indeed, C can also be seen as the complement in $\mathbb{A}^1$ of a finite number (possibly zero) of points and we can consider the closed embedding $\mu\colon C \to \mathbb{A}^2$ given by $x \mapsto (x, \frac{1}{f(x)})$, where $f \in \Bbbk[x]$ is a polynomial whose roots are exactly the removed points. Note that the image of μ is the curve of $\mathbb{A}^2$ defined by the equation $f(x)y = 1$.

By [DKR21, Proposition 1] the bracket width of $\mathrm{Vec}(\mathbb{A}^1)$ is one. By Theorem B the bracket width of $\mathrm{Vec}(\mathbb{A}^1 \setminus \Lambda)$ is smaller than or equal to the bracket width of $\mathrm{Vec}(\mathbb{A}^1)$ which is one. The proof follows. □

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