# On the annihilators of rational functions in the Lie algebra of derivations of $\mathbb{k}[x, y]$. 

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#### Abstract

Let $\mathbb{k}$ be an algebraically closed field of zero characteristic. The Lie algebra $W_{2}=W_{2}(\mathbb{k})$ of all $\mathbb{k}$-derivations of the polynomial ring $\mathbb{k}[x, y]$ naturally acts on the polynomial ring $\mathbb{k}[x, y]$ and also on the field of rational functions $\mathbb{k}(x, y)$. For a fixed rational function $u \in \mathbb{k}(x, y) \backslash \mathbb{k}$ we consider the set $\mathcal{A}_{W_{2}}(u)$ of all derivations $D \in W_{2}$ such that $D(u)=0$. We prove that $\mathcal{A}_{W_{2}}(u)$ is a free submodule of rank 1 of the $\mathbb{k}[x, y]$-module $W_{2}$. A description of the maximal abelian subalgebras as well of the centralizers of elements in the Lie algebra $\mathcal{A}_{W_{2}}(u)$ has been obtained.


## Introduction

Let $\mathbb{k}$ be a field of characteristic zero. The Lie algebra $W_{n}=W_{n}(\mathbb{k})$ of all $\mathbb{k}$-derivations of the polynomial ring $\mathbb{k}\left[x_{1}, \ldots, x_{n}\right]$ was studied by many authors from different points of view. Subalgebras of $W_{n}$ that are free $\mathbb{k}\left[x_{1}, \ldots, x_{n}\right]$-submodules of maximal rank in $W_{n}$, were studied by V. M. Buchstaber and D. V. Leykin in [2]. Using results of D. Jordan (4) one can point out some classes of simple subalgebras of $W_{n}$ that are also $\mathbb{k}\left[x_{1}, \ldots, x_{n}\right]$-submodules of the $\mathbb{k}\left[x_{1}, \ldots, x_{n}\right]$-module $W_{n}$. In 7 the centralizers of elements and the maximal abelian subalgebras of the algebra $s a_{2}(\mathbb{k})$ of all derivations $D \in W_{2}$ with zero divergence have been studied.

In this paper we study a class of subalgebras of the Lie algebra $W_{2}(\mathbb{k})$ over an algebraically closed field of characteristic zero. This class is determined by the natural action of the Lie algebra $W_{2}(\mathbb{k})$ on the field of rational functions $\mathbb{k}(x, y)$. Recall that every derivation $D \in W_{2}(\mathbb{k})$ of the ring $\mathbb{k}[x, y]$ can be uniquely extended to a derivation of the field $\mathbb{k}(x, y)$. It is natural to consider for a fixed rational function $u \in \mathbb{k}(x, y) \backslash \mathbb{k}$ the set $\mathcal{A}_{W_{2}}(u)$ of all derivations $D \in W_{2}$ such that $D(u)=0$. This set will be called the annihilator of $u$ in $W_{2}(\mathbb{k})$. It is a Lie subalgebra of $W_{2}(\mathbb{k})$ and at the same time a $\mathbb{k}[x, y]$-submodule of the $\mathbb{k}[x, y]$-module $W_{2}(\mathbb{k})$.

Using some results from [1], [5], [6], [8] we prove (Theorem (6) that for a rational function $u \in \mathbb{k}(x, y) \backslash \mathbb{k}$ its annihilator $\mathcal{A}_{W_{2}}(u)$ is a free submodule of rank 1 in the $\mathbb{k}[x, y]$-module $W_{2}$. We give also a free generator of this module. We describe the

[^0]centralizers of elements and the maximal abelian subalgebras of the Lie algebra $\mathcal{A}_{W_{2}}(u)$ (Theorems 9 and (12). It turned out that the algebra $\mathcal{A}_{W_{2}}(u)$ has completely different structure in the cases when $u$ is a polynomial and when $u$ is a rational function of the form $u=p / q$ with algebraically independent polynomials $p$ and $q$.

The notations used in the paper are standard. For a rational function $u \in \mathbb{k}(x, y)$ we denote by $\tilde{u}$ its generative rational function, i. e., a generator of the maximal subfield in $\mathbb{k}(x, y)$ of transcendence degree 1 that contains $u$. Recall (see [ $\mathbb{Z}]$ for details) that $\tilde{u}$ is defined uniquely up to linear fractional transformations

$$
\frac{\alpha \tilde{u}+\beta}{\gamma \tilde{u}+\delta}, \quad \alpha \delta-\beta \gamma \neq 0
$$

Note also that if $u$ is a polynomial there exists a polynomial generative function $\tilde{u} \in$ $\mathbb{k}[x, y]$. Recall that if a rational function or a polynomial is generative for itself, then it is called closed.

A derivation $D=P \frac{\partial}{\partial x}+Q \frac{\partial}{\partial y}$ will be called reduced if the polynomials $P$ and $Q$ are coprime, i.e. $\operatorname{gcd}(P, Q)=1$. For an arbitrary polynomial $u \in \mathbb{k}[x, u]$ we denote by $D_{u}$ the derivation of $\mathbb{k}(x, y)$ given by the rule $D_{u}(\varphi)=\operatorname{det} J(u, \varphi)=\frac{\partial u}{\partial x} \frac{\partial \varphi}{\partial y}-\frac{\partial u}{\partial y} \frac{\partial \varphi}{\partial x}$, i. e.,

$$
D_{u}=-\frac{\partial u}{\partial y} \frac{\partial}{\partial x}+\frac{\partial u}{\partial x} \frac{\partial}{\partial y}
$$

If $u$ possesses a polynomial generative function, then one can choose $\tilde{u}$ to be an irreducible polynomial. If $u$ does not have a polynomial generative function, then one can choose $\tilde{u}=p / q$ for some irreducible and algebraically independent polynomials $p$ and $q$ (see, for example, [1] or [8], Corollary 1).

## 1 On the structure of the $\mathbb{k}[x, y]$-module $\mathcal{A}_{W_{2}}(u)$.

Lemma 1. Let $D=P \frac{\partial}{\partial x}+Q \frac{\partial}{\partial y} \in W_{2}(\mathbb{k})$ be a reduced derivation. Then $D$ has a non-trivial kernel, i. e., Ker $D \supsetneq \mathbb{k}$, if and only if there exist non-zero polynomials $h, u \in \mathbb{k}[x, y], u \notin \mathbb{k}$, such that $h D=D_{u}$.
Proof. If $h D=D_{u}$, then $h D(u)=D_{u}(u)=\operatorname{det} J(u, u)=0$. As $h$ is different from zero, one concludes that $u$ belongs to the kernel of $D$.

Let now $D(u)=0$ for some non-constant polynomial $u$. The latter means $P \frac{\partial u}{\partial x}+$ $Q \frac{\partial u}{\partial y}=0$ and using that $P$ and $Q$ are coprime we obtain $\frac{\partial u}{\partial x}=h Q$ and $\frac{\partial u}{\partial y}=-h P$ for some polynomial $h$. Thus $h D=D_{u}$.
Lemma 2. Let $u \in \mathbb{k}(x, y) \backslash \mathbb{k}$ and let $\tilde{u}$ be its generative rational function. Then $\mathcal{A}_{W_{2}}(u)=\mathcal{A}_{W_{2}}(\tilde{u})$.
Proof. Since $\tilde{u}$ is a generative rational function for $u$, we obtain $u=F(\tilde{u})$ for some non-constant rational function $F(t) \in \mathbb{k}(t)$. Then for every derivation $D \in W_{2}$ one has

$$
D(u)=D(F(\tilde{u}))=F^{\prime}(\tilde{u}) \cdot D(\tilde{u})
$$

and using that $F^{\prime}(\tilde{u}) \neq 0$ we conclude that $D(u)=0$ if and only if $D(\tilde{u})=0$. This implies $\mathcal{A}_{W_{2}}(u)=\mathcal{A}_{W_{2}}(\tilde{u})$ and completes the proof.

Following [6] we assign to every irreducible polynomial $p \in \mathbb{k}[x, y]$ and every closed rational function $p / q$ with algebraically independent irreducible $p$ and $q$ reduced derivations $\delta_{p}$ and $\delta_{p, q}$ respectively. For an irreducible polynomial $p \in \mathbb{k}[x, y]$ the derivation $D_{p}$ may be written as $D_{p}=-\frac{\partial p}{\partial y} \frac{\partial}{\partial x}+\frac{\partial p}{\partial x} \frac{\partial}{\partial y}$. Let $h=\operatorname{gcd}\left(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}\right)$, put $P=-\frac{\partial p}{\partial y} / h$, $Q=\frac{\partial p}{\partial x} / h$, and denote $\delta_{p}=P \frac{\partial}{\partial x}+Q \frac{\partial}{\partial y}$. Note that $\operatorname{gcd}(P, Q)=1$ and the derivation $\delta_{p}$ is defined by the polynomial $p$ uniquely up to multiplication by a non-zero element from $\mathbb{k}$.

Analogously for a rational function $\varphi=p / q$ such that $p$ and $q$ are irreducible and algebraically independent polynomials we denote by $D_{p, q}$ the derivation defined from the formula

$$
D_{p, q}(f) \cdot d x \wedge d y=(q d p-p d q) \wedge d f
$$

One easily computes

$$
\begin{aligned}
(q d p-p d q) \wedge d f=q \cdot d p & \wedge d f-p \cdot d q \wedge d f= \\
& =\left[q\left(\frac{\partial p}{\partial x} \frac{\partial f}{\partial y}-\frac{\partial p}{\partial y} \frac{\partial f}{\partial x}\right)-p\left(\frac{\partial q}{\partial x} \frac{\partial f}{\partial y}-\frac{\partial q}{\partial y} \frac{\partial f}{\partial x}\right)\right] d x \wedge d y
\end{aligned}
$$

hence we obtain the equality

$$
D_{p, q}=\left(p \frac{\partial q}{\partial y}-q \frac{\partial p}{\partial y}\right) \frac{\partial}{\partial x}+\left(q \frac{\partial p}{\partial x}-p \frac{\partial q}{\partial x}\right) \frac{\partial}{\partial y}
$$

Let $P_{0}=p \frac{\partial q}{\partial y}-q \frac{\partial p}{\partial y}, Q_{0}=q \frac{\partial p}{\partial x}-p \frac{\partial q}{\partial x}$. Denote $h=\operatorname{gcd}\left(P_{0}, Q_{0}\right)$ and put

$$
\delta_{p, q}=\frac{1}{h} \cdot D_{p, q}
$$

Note again that $\delta_{p, q}$ is defined uniquely up to multiplication by a non-zero constant.
Lemma 3. (1) Let $p$ be an irreducible polynomial. Then $D_{p}(F(p))=0$ for every rational function $F \in \mathbb{k}(t)$.
(2) Let $p$ and $q$ be irreducible algebraically independent polynomials. Then

$$
D_{p, q}(p)=\operatorname{det}(J(p, q)) \cdot p, \quad D_{p, q}(q)=\operatorname{det}(J(p, q)) \cdot q
$$

(3) For every homogeneous polynomial $f(x, y)$ of degree $m$ it holds

$$
D_{p, q}(f(p, q))=m \operatorname{det} J(p, q) f(p, q)
$$

Proof. (1) As $D_{p}(p)=0$, we conclude that $D_{p}(F(p))=F^{\prime}(p) D_{p}(p)=0$.
(2) One computes

$$
\begin{aligned}
& D_{p, q}(p)=\left(p \frac{\partial q}{\partial y}-q \frac{\partial p}{\partial y}\right) \frac{\partial p}{\partial x}+\left(q \frac{\partial p}{\partial x}-p \frac{\partial q}{\partial x}\right) \frac{\partial p}{\partial y}= \\
&\left(\frac{\partial p}{\partial x} \frac{\partial q}{\partial y}-\frac{\partial p}{\partial y} \frac{\partial q}{\partial x}\right) p=\operatorname{det}(J(p, q)) \cdot p
\end{aligned}
$$

Analogous straightforward computation shows that $D_{p, q}(q)=\operatorname{det}(J(p, q)) \cdot q$.
(3) Let $f(x, y)=\sum_{i=0}^{m} a_{i} x^{i} y^{m-i}, a_{i} \in \mathbb{k}$, be a homogeneous polynomial of degree $m$ in variables $x$ and $y$. Then

$$
\begin{aligned}
D_{p, q}(f(p, q))= & D_{p, q}\left(\sum_{i=0}^{m} a_{i} p^{i} q^{m-i}\right)=\sum_{i=0}^{m} D_{p, q}\left(a_{i} p^{i} q^{m-i}\right)= \\
& \sum_{i=0}^{m} a_{i}\left(i p^{i-1} D_{p, q}(p) q^{m-i}+(m-i) q^{m-i-1} D_{p, q}(q) p^{i}\right)= \\
& \sum_{i=0}^{m} a_{i}\left(i p^{i} q^{m-i}+(m-i) q^{m-i} p^{i}\right) \operatorname{det}(J(p, q))=m \operatorname{det}(J(p, q)) f(p, q) .
\end{aligned}
$$

This proves the last part of the lemma.
For convenience let us introduce the following notations. Let $\varphi \in \mathbb{k}(x, y) \backslash \mathbb{k}$ be a non-constant rational function.

If $\varphi$ has a polynomial generative functions, then there exists an irreducible generative polynomial $p$ of $\varphi$. Put $\delta_{\varphi}:=\delta_{p}$.

If $\varphi$ does not have any polynomial generative function, we find a generative rational function of the form $p / q$ with irreducible and algebraically independent polynomials $p, q \in \mathbb{k}[x, y]$. Put in this case $\delta_{\varphi}:=\delta_{p, q}$.

Lemma 4. Let $D$ be a derivation of $\mathbb{k}[x, y]$ and $p, q \in \mathbb{k}[x, y]$ be two algebraically independent polynomials such that $D(p)=D(q)=0$. Then $D=0$.

Proof. We can consider $D$ as a derivation of the field $\mathbb{k}(x, y)$. Its kernel is an algebraically closed subfield of $\mathbb{k}(x, y)$ by Lemma 2.1 from [5]. Since $p$ and $q$ are algebraically independent, one concludes that $\operatorname{Ker} D=\mathbb{k}(x, y)$. Thus $D=0$. This completes the proof.

Lemma 5. Let $D_{1}, D_{2} \in W_{2}$ and let $D_{1}$ be a reduced derivation. If $u D_{1}+v D_{2}=0$ for some polynomials $u, v \in \mathbb{k}[x, y]$, then $v$ divides $u$ and $D_{2}=f D_{1}$ for $f=u / v \in \mathbb{k}[x, y]$.
Proof. Let $D_{1}=P_{1} \frac{\partial}{\partial x}+Q_{1} \frac{\partial}{\partial y}, D_{2}=P_{2} \frac{\partial}{\partial x}+Q_{2} \frac{\partial}{\partial y}$. Then $u P_{1}+v P_{2}=0$ and $u Q_{1}+v Q_{2}=$ 0 . From these equalities it follows that $v$ divides $u P_{1}$ and $u Q_{1}$. Since the polynomials $P_{1}$ and $Q_{1}$ are coprime, $u$ is divisible by $v$ and we obtain $P_{2}=f Q_{1}, Q_{2}=f Q_{1}$ for $f=u / v$. Hence $D_{2}=f D_{1}$.

Theorem 6. For an arbitrary rational function $\varphi \in \mathbb{k}(x, y) \backslash \mathbb{k}$ its annihilator $\mathcal{A}_{W_{2}}(\varphi)$ is a free submodule of rank 1 of the $\mathbb{k}[x, y]$-module $W_{2}(\mathbb{k})$. As a free generator of the submodule $\mathcal{A}_{W_{2}}(\varphi)$ one can choose the derivation $\delta_{\varphi}$.
Proof. Let $\delta_{\varphi}=P_{0} \frac{\partial}{\partial x}+Q_{0} \frac{\partial}{\partial y}$. Then the polynomials $P_{0}$ and $Q_{0}$ are coprime by construction. Let us take an arbitrary derivation $D=P \frac{\partial}{\partial x}+Q \frac{\partial}{\partial y}$ from $\mathcal{A}_{W_{2}}(\varphi)$. We shall show that $D=h \delta_{\varphi}$ for some polynomial $h \in \mathbb{k}[x, y]$.

Consider the case when $\varphi$ possesses a polynomial generative function $p$ ．Then by Lemma 2 we get $\mathcal{A}_{W_{2}}(\varphi)=\mathcal{A}_{W_{2}}(p)$ ．Therefore，$D(p)=0$ and by Lemma 1 we conclude that there exists a polynomial $h_{0}$ such that $\frac{\partial p}{\partial x}=h_{0} Q$ and $\frac{\partial p}{\partial y}=-h_{0} P$ ．This means that $D_{p}=h_{0} D$ ．By definition of $\delta_{p}$ there is a polynomial $h_{1} \in \mathbb{k}[x, y]$ such that $D_{p}=h_{1} \delta_{p}$ and therefore $h_{0} D-h_{1} \delta_{p}=0$ ．Since $\delta_{p}$ is a reduced derivation，we have by Lemma 5 that $D=h \delta_{p}$ for some polynomial $h \in \mathbb{k}[x, y]$ ．

Let us consider now the case when $\varphi$ does not have any polynomial generative function．In this case $\delta_{\varphi}=\delta_{p, q}$ for some irreducible and algebraically independent polynomials $p$ and $q$ such that $p / q$ is a generative rational function for $\varphi$ ．By Lemma 2 $\mathcal{A}_{W_{2}}(\varphi)=\mathcal{A}_{W_{2}}(p / q)$ ．Since $D_{p, q}(p / q)=0$ ，from the definition of $\delta_{p, q}$ it follows that $\delta_{p, q}(p / q)=0$ ．Therefore，$\delta_{p, q}$ belongs to $\mathcal{A}_{W_{2}}(p / q)=\mathcal{A}_{W_{2}}(\varphi)$ ．

Let $D=P \frac{\partial}{\partial x}+Q \frac{\partial}{\partial y}$ be an arbitrary non－zero derivation from $\mathcal{A}_{W_{2}}(\varphi)$ ．Since $D(\varphi)=$ 0 implies $D(p / q)=0$ and since $D(p / q)=\frac{D(p) q-p D(q)}{q^{2}}$ ，we conclude $D(p) q-p D(q)=0$ ． As the polynomials $p$ and $q$ are coprime，we obtain that $D(p)=\lambda p$ and $D(q)=\lambda q$ for some $\lambda \in \mathbb{k}[x, y]$ ．Denote for convenience $\mu=\operatorname{det} J(p, q)$ ．Then by Lemma 3 $h \delta_{p, q}(p)=D_{p, q}(p)=\mu p$ and $h \delta_{p, q}(q)=D_{p, q}(q)=\mu q$ ．As the polynomials $p$ and $q$ lie in the kernel of the derivation $\lambda h \delta_{p, q}-\mu D$ we have by Lemma $⿴ 囗 ⿱ 一 一{ }^{-1}$ that $\lambda h \delta_{p, q}-\mu D=0$ ． The derivation $\delta_{p, q}$ is reduced by construction，so by Lemma 5 we obtain $D=h \delta_{p, q}$ for some polynomial $h$ ．We proved that every derivation $D \in \mathcal{A}_{W_{2}}(\varphi)$ is of the form $h \delta_{\varphi}$ for some polynomial $h \in \mathbb{k}[x, y]$ ．This shows that $\mathcal{A}_{W_{2}}(\varphi)$ is a free $\mathbb{k}[x, y]$－module of rank 1 with the generator $\delta_{\varphi}$ ．

## 2 On centralizers of elements and maximal abelian subalgebras of the Lie algebra $\mathcal{A}_{W_{2}}(\varphi)$ ．

Definition 7．Let $p(x, y)$ be an irreducible polynomial．A polynomial $f(x, y)$ will be called $p$－free if $f(x, y)$ is not divisible by any polynomial in $p(x, y)$ of positive degree．

It is clear that for every polynomial $g(x, y)$ there exists a $p$－free polynomial $\bar{g}(x, y)$ such that $g=\bar{g} \cdot h(p)$ for some $h \in \mathbb{k}[t]$ ．Note that $\bar{g}$ is determined by the polynomial $g$ uniquely up to multiplication by a non－zero constant．
Lemma 8．Let $u \in \mathbb{k}(x, y) \backslash \mathbb{k}$ be a non－constant rational function with a polynomial generative function and $p(x, y)$ be an irreducible generative polynomial for $u(x, y)$ ．Then $C_{\mathcal{A}_{W_{2}}(u)}\left(g \delta_{p}\right)=C_{\mathcal{A}_{W_{2}}(u)}\left(\bar{g} \delta_{p}\right)$ for any $g \in \mathbb{k}[x, y]$ ，where $\bar{g}$ is the $p$－free polynomial corresponding to $g$ ．
Proof．Let $g=\bar{g} h$ ，where $h \in \mathbb{k}[p]$ ．Take an arbitrary derivation $D$ from $C_{\mathcal{A}_{W_{2}}(u)}\left(g \delta_{p}\right)$ ． By Theorem 6 $D$ is of the form $D=f \delta_{p}$ for some polynomial $f$ ．Since $\left[f \delta_{p}, g \delta_{p}\right]=0$ ， we have

$$
0=\left[f \delta_{p}, g \delta_{p}\right]=\left[f \delta_{p}, h \bar{g} \delta_{p}\right]=f \delta_{p}(h) \bar{g} \delta_{p}+h\left[f \delta_{p}, \bar{g} \delta_{p}\right] .
$$

Since $\delta_{p}(h)=0$ and $h \neq 0$ ，we obtain from the last equalities that $D=f \delta_{p} \in$ $C_{\mathcal{A}_{W_{2}}(u)}\left(\bar{g} \delta_{p}\right)$ ．Conversely，let $D \in C_{\mathcal{A}_{W_{2}}(u)}\left(\bar{g} \delta_{p}\right)$ ．Using the same notations，write now $D=f \delta_{p}$ for some $f \in \mathbb{k}[x, y]$ ．Then $\left[f \delta_{p}, \bar{g} \delta_{p}\right]=0$ ．But then

$$
\left[f \delta_{p}, g \delta_{p}\right]=\left[f \delta_{p}, \bar{g} h \delta_{p}\right]=f \delta_{p}(h) \bar{g} \delta_{p}+h\left[f \delta_{p}, \bar{g} \delta_{p}\right]=0
$$

Therefore, the derivation $f \delta_{p}$ belongs to $C_{\mathcal{A}_{W_{2}}(u)}\left(g \delta_{p}\right)$ if and only if it belongs to $C_{\mathcal{A}_{W_{2}}(u)}\left(\bar{g} \delta_{p}\right)$. This proves the required statement.

Theorem 9. Let $u \in \mathbb{k}(x, y) \backslash \mathbb{k}$ be a non-constant rational function that possesses a polynomial generative function. Let $p(x, y)$ be an irreducible generative polynomial for $u(x, y)$. Then
(1) the centralizer of an arbitrary element form $\mathcal{A}_{W_{2}}(u)$ in the Lie algebra $\mathcal{A}_{W_{2}}(u)$ equals $\bar{f} \mathbb{k}[p] \delta_{p}$, where $\bar{f}$ is a p-free polynomial corresponding to $f$;
(2) maximal abelian subalgebras of $\mathcal{A}_{W_{2}}(u)$ and only they are of the form $\bar{f} \mathbb{k}[p] \delta_{p}$, where $\bar{f}$ is a $p$-free polynomial.

Proof. (1) Since by Lemma $8 C_{\mathcal{A}_{W_{2}}(u)}\left(f \delta_{p}\right)=C_{\mathcal{A}_{W_{2}}(u)}\left(\bar{f} \delta_{p}\right)$, we can assume without loss of generality that $f=\bar{f}$. Take an arbitrary element $g \delta_{p}$ from $C_{\mathcal{A}_{W_{2}}(u)}\left(f \delta_{p}\right)$. Denote by $\bar{g}$ a $p$-free polynomial corresponding to $g$. Then $g=\bar{g} \cdot h_{0}$, where $h_{0}=h_{0}(p)$ is a polynomial in $p$. By Lemma 8 it holds $\left[f \delta_{p}, \bar{g} \delta_{p}\right]=0$ and therefore $\delta_{p}(f) \bar{g}-f \delta_{p}(\bar{g})=0$. This relation yields the equality $\delta_{p}(\bar{g} / f)=0$. As $D_{p}=\lambda \delta_{p}$ for some polynomial $\lambda$, we have $D_{p}(\bar{g} / f)=0$. Since $D_{p}(\bar{g} / f)=\operatorname{det}(J(p, \bar{g} / f))$, the latter implies that the rational functions $p$ and $\bar{g} / f$ are algebraically dependent (see for example [3], Ch. III, $\S 7$, Th. III or [8], Lemma 1). As the polynomial $p$ is closed (it is irreducible), the function $\bar{g} / f$ belongs to the field $\mathbb{k}(p)$. This means that the rational function $\bar{g} / f$ can be written in the form $\bar{g} / f=u(p) / v(p)$ for some coprime polynomials $u, v \in \mathbb{k}[t]$. From the last equality it follows $\bar{g} v(p)=f u(p)$. As the polynomials $f$ and $\bar{g}$ are both $p$-free, we have $\bar{g}=c f$ for some $c \in \mathbb{k}^{\times}$(a $p$-free polynomial corresponding to the polynomial $\bar{g} v(p)=f u(p)$ is determined uniquely up to nonzero constant multiplier). Hence $g=h_{0} \bar{g}=h_{0} c f$, where $h_{0} c \in \mathbb{k}[p]$. We proved the inclusion $C_{\mathcal{A}_{W_{2}}(u)}\left(f \delta_{p}\right) \subseteq \mathbb{k}[p] \cdot f \delta_{p}$.

Since for every polynomial $r \in \mathbb{k}[t]$ we have

$$
\left[r(p) f \delta_{p}, f \delta_{p}\right]=\left(r(p) f \delta_{p}(f)-f \delta_{p}(r(p) f)\right) \delta_{p}=\left(r(p) f \delta_{p}(f)-f r(p) \delta_{p}(f)\right) \delta_{p}=0
$$

it holds also $\mathbb{k}[p] \cdot f \delta_{p} \subseteq C_{\mathcal{A}_{W_{2}}(u)}\left(f \delta_{p}\right)$. We proved the equality $\mathbb{k}[p] \cdot f \delta_{p}=C_{\mathcal{A}_{W_{2}}(u)}\left(f \delta_{p}\right)$ for $p$-free $f$. This proves the first part of the theorem.
(2) Let $M$ be a maximal abelian subalgebra in $\mathcal{A}_{W_{2}}(u)$ and let $f \delta_{p}$ be an arbitrary non-zero element of $M$. Then $M \subseteq C_{\mathcal{A}_{W_{2}}(u)}\left(f \delta_{p}\right)$ and by the part (1) of this theorem $C_{\mathcal{A}_{W_{2}}(u)}\left(f \delta_{p}\right)=\mathbb{k}[p] \cdot \bar{f} \delta_{p}$. Since for arbitrary polynomials $F, G \in \mathbb{k}[t]$ it holds

$$
\begin{aligned}
{\left[F(p) \bar{f} \delta_{p}, G(p) \bar{f} \delta_{p}\right] } & =\left(F(p) \bar{f} \delta_{p}(G(p) \bar{f})-G(p) \bar{f} \delta_{p}(F(p) \bar{f})\right) \cdot \delta_{p} \\
& =\left(F(p) G(p) \bar{f} \delta_{p}(\bar{f})-F(p) G(p) \bar{f} \delta_{p}(\bar{f})\right) \cdot \delta_{p}=0
\end{aligned}
$$

one sees that $\mathbb{k}[p] \cdot \bar{f} \delta_{p}$ is an abelian algebra. The maximality of $M$ implies $M=\mathbb{k}[p] \cdot \bar{f} \delta_{p}$. Conversely, the subalgebras of the form $\mathbb{k}[p] \cdot \bar{f} \delta_{p}$ are abelian Lie algebras for any $p$ free polynomial $\bar{f}$. As every element commuting with $\bar{f} \delta_{p}$ belongs by definition to the centralizer $C_{\mathcal{A}_{W_{2}}(u)}\left(\bar{f} \delta_{p}\right)=\mathbb{k}[p] \bar{f} \delta$, one sees that all such subalgebras are maximal abelian. This completes the proof of the theorem.

Definition 10. Let $p$ and $q$ be algebraically independent irreducible polynomials from the ring $\mathbb{k}[x, y]$. A polynomial $f(x, y) \in \mathbb{k}[x, y]$ will be called $p$ - $q$-free if $f$ is not divisible by any homogeneous polynomial in $p$ and $q$ of positive degree.

It is clear that for every polynomial $f$ there exists a $p-q$-free polynomial $\bar{f}$ such that $f=\bar{f} h$ for some homogeneous in $p$ and $q$ polynomial $h$. We denote by $\mathbb{k}[p, q]_{m}$ the vector space of all polynomials $f(p, q)$, where $f(x, y)$ is a homogeneous polynomial of degree $m$.

Lemma 11. Let $p, q \in \mathbb{k}[x, y]$ be irreducible algebraically independent polynomials, $\alpha_{1}, \alpha_{2}, \beta_{1}, \beta_{2} \in \mathbb{k}$. Then the polynomials $\alpha_{1} p+\beta_{1} q$ and $\alpha_{2} p+\beta_{2} q$ are either linearly dependent (over $\mathbb{k}$ ) or coprime. In the first case $\left(\alpha_{1}: \beta_{1}\right)=\left(\alpha_{2}: \beta_{2}\right)$ as points in $\mathbb{P}_{1}(\mathbb{k})$. In the second case $\left(\alpha_{1}: \beta_{1}\right) \neq\left(\alpha_{2}: \beta_{2}\right)$.
Proof. Since $p$ and $q$ are algebraically independent, they are also linearly independent. If the polynomials $r_{1}=\alpha_{1} p+\beta_{1} q$ and $r_{2}=\alpha_{2} p+\beta_{2} q$ are linearly dependent then clearly $\left(\alpha_{1}: \beta_{1}\right)=\left(\alpha_{2}: \beta_{2}\right)$. Let now $r_{1}$ and $r_{2}$ be linearly independent. Then $\operatorname{det}\left(\begin{array}{c}\alpha_{1} \beta_{1} \\ \alpha_{2} \\ \beta_{2}\end{array}\right) \neq 0$. We can write down $p=a_{1} r_{1}+b_{1} r_{2}$ and $q=a_{2} r_{1}+b_{2} r_{2}$ for some $a_{1}, a_{2}, b_{1}, b_{2} \in \mathbb{k}$. It is clear that any common divisor of $r_{1}$ and $r_{2}$ is a common divisor of $p$ and $q$. Since $p$ and $q$ are coprime, we conclude that $r_{1}$ and $r_{2}$ are coprime as well. In this case $\left(\alpha_{1}: \beta_{1}\right) \neq\left(\alpha_{2}: \beta_{2}\right)$.

Theorem 12. Let $\varphi$ be a rational function that does not possess a polynomial generative function. Let $p$ and $q$ be algebraically independent and irreducible polynomials such that $p / q$ is a generative rational function for $\varphi$. Then
(1) for an arbitrary element $f \delta_{p, q} \in \mathcal{A}_{W_{2}}(\varphi)$ its centralizer in $\mathcal{A}_{W_{2}}(\varphi)$ coincides with $\mathbb{k}[p, q]_{m} \cdot \bar{f} \delta_{p, q}$, where $\bar{f}$ is a p-q-free polynomial such that $f=\bar{f} \cdot h$ for some polynomial $h$ homogeneous of degree $m$ in $p$ and $q$;
(2) every maximal abelian subalgebra from $\mathcal{A}_{W_{2}}(\varphi)$ is of the form $\mathbb{k}[p, q]_{m} \bar{f} \delta_{p, q}$ for some integer $m$ and $p$ - $q$-free polynomial $\bar{f}$. Every subalgebra of the type $\mathbb{k}[p, q]_{m} \bar{f} \delta_{p, q}$ is maximal abelian. In particular all maximal abelian subalgebras of $\mathcal{A}_{W_{2}}(\varphi)$ are finite dimensional.

Proof. First of all note that one can choose for $\varphi$ a generative rational function of the form $p / q$ with irreducible and algebraically independent polynomials $p$ and $q$ by Corollary 1 from [8].
(1) By Lemma $2 \mathcal{A}_{W_{2}}(\varphi)=\mathcal{A}_{W_{2}}(p / q)$. By Theorem 6 every derivation from $\mathcal{A}_{W_{2}}(\varphi)$ may be written as $f \delta_{p, q}$ for some polynomial $f \in k[x, y]$.

Take an arbitrary element $g \delta_{p, q} \in C_{\mathcal{A}_{W_{2}}(\varphi)}\left(f \delta_{p, q}\right)$. Then $\left[g \delta_{p, q}, f \delta_{p, q}\right]=0$ and hence $g \delta_{p, q}(f)-f \delta_{p, q}(g)=0$. Therefore, $\delta_{p, q}(f / g)=0$ and also $D_{p, q}(f / g)=0$ (recall that $D_{p, q}=\lambda \delta_{p, q}$ for some $\left.\lambda \in \mathbb{k}[x, y]\right)$. Since $D_{p, q}(f / g)=\operatorname{det}(J(p / q, f / g))$, the latter implies that the rational functions $p / q$ and $f / g$ are algebraically dependent (see for example [3], Ch.III, §7, Th. III or [8], Lemma 1). As $p / q$ is a closed rational function, we see that $f / g \in \mathbb{k}(p / q)$. Therefore,

$$
\frac{f}{g}=\frac{F\left(\frac{p}{q}\right)}{G\left(\frac{p}{q}\right)}
$$

for some coprime polynomials $F, G \in \mathbb{k}[t]$.
By our assumption the field $\mathbb{k}$ is algebraically closed, so we can decompose $F$ and $G$ into linear factors, say

$$
F(t)=c_{1}\left(t-\lambda_{1}\right) \ldots\left(t-\lambda_{k}\right), \quad G(t)=c_{2}\left(t-\mu_{1}\right) \ldots\left(t-\mu_{l}\right), \quad c_{1}, c_{2}, \lambda_{i}, \mu_{j} \in \mathbb{k} .
$$

Then

$$
\frac{f}{g}=\frac{F\left(\frac{p}{q}\right)}{G\left(\frac{p}{q}\right)}=\frac{c_{1}\left(p-\lambda_{1} q\right) \ldots\left(p-\lambda_{k} q\right)}{c_{2}\left(p-\mu_{1} q\right) \ldots\left(p-\mu_{l} q\right)} \cdot q^{l-k}
$$

and hence

$$
g=\frac{c_{2}\left(p-\mu_{1} q\right) \ldots\left(p-\mu_{l} q\right)}{c_{1}\left(p-\lambda_{1} q\right) \ldots\left(p-\lambda_{k} q\right)} \cdot q^{k-l} \cdot f
$$

Write $f=\bar{f} \cdot h$ for a $p$ - $q$-free polynomial $\bar{f}$ and a polynomial $h$ homogeneous of degree $m$ in $p$ and $q$. It is known that the polynomial $h$ can be decomposed into the product

$$
h=\left(\alpha_{1} p-\beta_{1} q\right) \ldots\left(\alpha_{m} p-\beta_{m} q\right) .
$$

with some $\alpha_{i}, \beta_{j} \in \mathbb{k}$. As $f=\bar{f} h$, we can finally write

$$
g=\frac{c_{2}\left(p-\mu_{1} q\right) \ldots\left(p-\mu_{l} q\right)\left(\alpha_{1} p-\beta_{1} q\right) \ldots\left(\alpha_{m} p-\beta_{m} q\right)}{c_{1}\left(p-\lambda_{1} q\right) \ldots\left(p-\lambda_{k} q\right)} \cdot q^{k-l} \cdot \bar{f} .
$$

The polynomial $\bar{f}$ is not divisible by homogeneous nonconstant polynomials in $p$ and $q$, so using Lemma 11 we conclude that the rational function

$$
h_{1}=\frac{c_{2}\left(p-\mu_{1} q\right) \ldots\left(p-\mu_{l} q\right)\left(\alpha_{1} p-\beta_{1} q\right) \ldots\left(\alpha_{m} p-\beta_{m} q\right)}{c_{1}\left(p-\lambda_{1} q\right) \ldots\left(p-\lambda_{k} q\right)} \cdot q^{k-l}
$$

must be a polynomial, i. e., all factors of its denominator must occur as factors in the numerator. It is obvious that $h_{1}$ is a homogeneous polynomial of degree $l+m+(k-$ $l)-k=m$ in $p$ and $q$.

We proved that $g=h_{1} \cdot \bar{f}$, hence $g \delta_{p, q}=h_{1} \cdot \bar{f} \delta_{p, q} \in \mathbb{k}[p, q]_{m} \cdot \bar{f} \delta_{p, q}$. Therefore, $C_{\mathcal{A}_{W_{2}}(\varphi)}\left(f \delta_{p, q}\right) \subseteq \mathbb{k}[p, q]_{m} \cdot \bar{f} \delta_{p, q}$.

For every polynomial $h_{2}$ homogeneous of degree $m$ in $p$ and $q$ applying Lemma 3, 3 ), one obtains

$$
\begin{aligned}
{\left[h_{2} \bar{f} \delta_{p, q}, f \delta_{p, q}\right] } & =\left[h_{2} \bar{f} \delta_{p, q}, h \bar{f} \delta_{p, q}\right]=\left(\bar{f} \delta_{p, q}\left(h_{2}\right) h-h_{2} \bar{f} \delta_{p, q}(h)\right) \bar{f} \delta_{p, q}= \\
& =\left(m \operatorname{det} J(p, q) h_{2} h-h_{2} m \operatorname{det} J(p, q) h\right) \bar{f}^{2} \delta_{p, q}=0 .
\end{aligned}
$$

Therefore, $\mathbb{k}[p, q]_{m} \cdot \bar{f} \delta_{p, q} \subseteq C_{\mathcal{A}_{W_{2}}(\varphi)}\left(f \delta_{p, q}\right)$ and we obtain the equality

$$
C_{\mathcal{A}_{W_{2}}(\varphi)}\left(f \delta_{p, q}\right)=\mathbb{k}[p, q]_{m} \cdot \bar{f} \delta_{p, q} .
$$

This proves the first part of the theorem.
(2) This part can be proved similarly to the part 2) of Theorem 9 by replacing the set $\mathbb{k}[p]$ by $\mathbb{k}[p, q]_{m}$.

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