WHEN IS THE AUTOMORPHISM GROUP OF AN AFFINE VARIETY LINEAR?

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Abstract. Let $\mathrm{Aut}_{\mathrm{alg}}(X)$ be the subgroup of the group of regular automorphisms $\mathrm{Aut}(X)$ of an affine algebraic variety X generated by all connected algebraic subgroups. We prove that if $\dim X \geq 2$ and if $\mathrm{Aut}_{\mathrm{alg}}(X)$ is "rich enough", $\mathrm{Aut}_{\mathrm{alg}}(X)$ is not linear, i.e., it cannot be embedded into $\mathrm{GL}_n(\mathbb{K})$, where $\mathbb{K}$ is an algebraically closed field of characteristic zero. Moreover, $\mathrm{Aut}(X)$ is isomorphic to an algebraic group as an abstract group only if the connected component of $\mathrm{Aut}(X)$ is either the algebraic torus or a direct limit of commutative unipotent groups. Finally, we prove that for an uncountable $\mathbb{K}$ the group of birational transformations of X cannot be isomorphic to the group of automorphisms of an affine variety if X is endowed with a rational action of a positive-dimensional linear algebraic group.

1. Introduction

In this paper we work over algebraically closed field $\mathbb{K}$ of characteristic zero, and X always denotes an irreducible affine variety. It is well-known that the automorphism group of an affine variety may be very large. For example, the automorphism group $\mathrm{Aut}(\mathbb{A}^2)$ of the affine plane $\mathbb{A}^2$ contains a free product of two polynomial rings in one variable. Consequently, $\mathrm{Aut}(\mathbb{A}^2)$ is infinite-dimensional and cannot be given a structure of an algebraic group. Moreover, it is shown in [3, Proposition 2.3] that $\mathrm{Aut}(\mathbb{A}^2)$ is not linear, i.e. $\mathrm{Aut}(\mathbb{A}^2)$ cannot be embedded into the general linear group $\mathrm{GL}_n(\mathbb{K})$ as an abstract group. The first main result of the present note is a generalization of this statement to a big family of affine varieties.

It is well-known (Proposition 2.3) that the automorphism group $\mathrm{Aut}(X)$ has a structure of an *ind-group* (see Section 2.2 for the definition) and if $\dim X \geq 2$, $\mathrm{Aut}(X)$ is infinite-dimensional unless $\mathrm{Aut}(X)$ is a countable extension of the algebraic torus. But even if the automorphism group $\mathrm{Aut}(X)$ is infinite-dimensional it may happen that $\mathrm{Aut}(X)$ embeds into $\mathrm{GL}_n(\mathbb{K})$. For exmaple, [10, Example 6.14] shows that there is an affine surface S such that $\mathrm{Aut}(S)$ is isomorphic to the polynomial ring in one variable $\mathbb{K}[t]$ and as an abstract additive group $\mathrm{Aut}(S)$ is isomorphic to the additive group of the base field and hence embeds into $\mathrm{GL}_2(\mathbb{K})$. However, if $\mathrm{Aut}(X)$ is rich enough, $\mathrm{Aut}(X)$ cannot be embedded into $\mathrm{GL}_n(\mathbb{K})$. More precisely, we prove the following statement.

We denote the additive and multiplicative group of the field $\mathbb{K}$ by $\mathbb{G}_a$ and $\mathbb{G}_m$ respectively. For a given affine variety X we denote by $\mathrm{Aut}_{\mathrm{alg}}(X)$ the subgroup of $\mathrm{Aut}(X)$ generated by all connected algebraic subgroups.

Theorem 1.1. *Assume dimension of X is ≥ 2 and X is not isomorphic to $\mathbb{A}^1 \times Y$, where Y is an affine variety that admits no faithful action of positive-dimensional connected algebraic group. If $\mathrm{Aut}_{\mathrm{alg}}(X)$ is non-commutative and contains a copy of $\mathbb{G}_m$, then $\mathrm{Aut}_{\mathrm{alg}}(X)$ cannot be embedded into $\mathrm{GL}_n(\mathbb{K})$.*

If X is isomorphic to $\mathbb{A}^1 \times Y$ for some affine variety Y, where Y does not admit a faithful action of $\mathbb{G}_m$ or $\mathbb{G}_a$, then $\mathrm{Aut}_{\mathrm{alg}}(X)$ is isomorphic to $\mathcal{O}(Y)^* \ltimes \mathcal{O}(Y)^+$ which can be embedded into $\mathbb{K}(Y)^* \ltimes \mathbb{K}(Y)^+$ which in turn embeds into $\overline{\mathbb{K}(Y)}^* \ltimes \overline{\mathbb{K}(Y)}^+ \simeq \mathbb{K}^* \ltimes \mathbb{K}^+$.

If X admits no $\mathbb{G}_m$-action, but admits two non-commuting $\mathbb{G}_a$-actions, $\mathrm{Aut}(X)$ can be embedded into $\mathrm{GL}_n(\mathbb{K})$. For example, there exists an affine surface X (see [1, Example 4.1.3]) that has an automorphism group $\mathrm{Aut}(X) = \mathrm{Aut}_{\mathrm{alg}}(X) \simeq \mathbb{K}[x] * \mathbb{K}[y]$ which is linear by [11, Theorem] as additive groups $\mathbb{K}[x] \simeq \mathbb{K}[y]$ are isomorphic as abstract groups to $\mathbb{G}_a$.

The main idea of the proof of Theorem 1.1 is to find a subgroup of $\mathrm{Aut}_{\mathrm{alg}}(X)$ that is isomorphic to a direct limit of subgroups isomorphic to $\mathbb{G}_m \ltimes \mathbb{G}_a^r$, $r \geq 2$, where $\mathbb{G}_a^r$ is a direct product of root subgroups with respect to $\mathbb{G}_m$ of different weights, and show that such a group cannot be embedded into $\mathrm{GL}_n(\mathbb{K})$.

As an application of Theorem 1.1 we obtain that the automorphism group $\mathrm{Aut}(X)$ is isomorphic to a linear algebraic group as an abstract group if and only if the connected component $\mathrm{Aut}^\circ(X) \subset \mathrm{Aut}(X)$ is commutative (see Theorem 4.2). Moreover, in this case $\mathrm{Aut}^\circ(X)$ is either the algebraic torus or a direct limit of commutative unipotent groups.

We denote by $\mathrm{Bir}(X)$ the group of birational transformations of X. It is well-known that such a group may be very large. For example the Cremona group $\mathrm{Bir}(\mathbb{A}^n) = \mathrm{Bir}(\mathbb{P}^n)$ for $n > 1$ is known to be very big, in particular, much larger than $\mathrm{Aut}(\mathbb{A}^n)$. Proposition 5.1 shows that the Cremona group $\mathrm{Bir}(\mathbb{A}^n) = \mathrm{Bir}(\mathbb{P}^n)$, $n > 0$, is not isomorphic to the automorphism group of any affine variety. Moreover, if $\mathrm{Bir}(X)$ is "rich enough", $\mathrm{Bir}(X)$ is also not isomorphic to the automorphism group of any affine variety. More precisely, we prove the following result.

Theorem 1.2. *Assume $\mathbb{K}$ is uncountable and X, Y are affine irreducible algebraic varieties. Assume X is endowed with a rational action of a positive-dimensional linear algebraic group. Then the group of birational transformations* $\mathrm{Bir}(X)$ *is not isomorphic to* $\mathrm{Aut}(Y)$.

Acknowledgements: I would like to thank Ievgen Makedonskyi for useful comments.

2. Preliminaries

2.1. **Derivations and group actions.** Recall that X is an irreducible affine algebraic variety. A derivation δ is called *locally finite* if it acts locally finitely on $\mathcal{O}(X)$, i.e., for any function $f \in \mathcal{O}(X)$ there is a finite-dimensional vector subspace $W \subset \mathcal{O}(X)$ such that $f \in W$ and W is stable under action of δ. A derivation $\delta \in \mathrm{Der}(\mathcal{O}(X))$ is called *locally nilpotent* if for any function $f \in \mathcal{O}(X)$ there exists $k \in \mathbb{N}$ (which depends on f) such that $\delta^k(f) = 0$. Note that there is a one-to-one correspondence between locally nilpotent derivations on $\mathcal{O}(X)$ and $\mathbb{G}_a$-actions on X given by the map $\delta \mapsto \{t \mapsto \exp(t\delta)\}$.

An element $u \in \mathrm{Aut}(X)$ is called *unipotent* if $u = \exp(\partial)$ for some locally nilpotent derivation ∂.

2.2. **Ind-groups.** The notion of an ind-group goes back to Shafarevich who called it an infinite dimensional algebraic group (see [14]). We refer to [5] for basic notions in this context.

Definition 2.1. By an affine *ind-variety* we understand an injective limit $V = \varinjlim V_i$ of an ascending sequence $V_0 \hookrightarrow V_1 \hookrightarrow V_2 \hookrightarrow \ldots$ such that the following holds:

(1) $V = \bigcup_{k \in \mathbb{N}} V_k$;
(2) each V_k is an affine algebraic variety;
(3) for all $k \in \mathbb{N}$ the embedding $V_k \hookrightarrow V_{k+1}$ is closed in the Zariski topology.

For simplicity we will call an affine ind-variety simply an ind-variety.

An ind-variety V has a natural *topology*: a subset $S \subset V$ is called closed, resp. open, if $S_k := S \cap V_k \subset V_k$ is closed, resp. open, for all $k \in \mathbb{N}$. A closed subset $S \subset V$ has a natural structure of an ind-variety and is called an ind-subvariety.

A set theoretical product of ind-varieties admits a natural structure of an ind-variety. A *morphism* between ind-varieties $V = \bigcup_m V_m$ and $W = \bigcup_l W_l$ is a map $\phi : V \to W$ such that for every $m \in \mathbb{N}$ there is an $l \in \mathbb{N}$ such that $\phi(V_m) \subset W_l$ and that the induced map $V_m \to W_l$ is a morphism of algebraic varieties. This allows us to give the following definition.

Definition 2.2. An ind-variety H is said to be an *ind-group* if the underlying set H is a group such that the map $H \times H \to H$, defined by $(g, h) \mapsto gh^{-1}$, is a morphism of ind-varieties.

A *closed subgroup* G of H is a subgroup that is at the same time a closed subset. In this case G is again an ind-group with respect to the induced ind-variety structure. A closed subgroup G of an ind-group $H = \varinjlim H_i$ is called an *algebraic subgroup* if G is contained in H_i for some i.

The next result can be found in [5, Section 5].

Proposition 2.3. *Let X be an affine variety. Then* $\mathrm{Aut}(X)$ *has the structure of an ind-group such that a regular action of an algebraic group H on X induces an ind-group homomorphism $H \to$* $\mathrm{Aut}(X)$*.*

2.3. **Root subgroups.** In this section we describe *root subgroups* of $\mathrm{Aut}(X)$ for a given affine variety X with respect to a subtorus.

Definition 2.4. Let T be a subtorus in $\mathrm{Aut}(X)$, i.e. a closed algebraic subgroup isomorphic to a torus. A closed algebraic subgroup $U \subset \mathrm{Aut}(X)$ isomorphic to $\mathbb{G}_a$ is called a *root subgroup* with respect to T if the normalizer of U in $\mathrm{Aut}(X)$ contains T.

Since $\mathbb{G}_a$ contains no non-trivial closed normal subgroups, every non-trivial regular action is faithful. Hence, such an algebraic subgroup U corresponds a non-trivial *normalized* $\mathbb{G}_a$-action on X, i.e. a $\mathbb{G}_a$-action on X whose image in $\mathrm{Aut}(X)$ is normalized by T.

Assume $U \subset \mathrm{Aut}(X)$ is a root subgroup with respect to T. Since T normalizes U, we can define an action $\varphi \colon T \to \mathrm{Aut}(U)$ of T on U given by $t.u = t \circ u \circ t^{-1}$ for all $t \in T$ and $u \in U$. Moreover, since $\mathrm{Aut}(U) \simeq \mathbb{G}_m$, such an action corresponds to a character of the torus $\chi \colon T \to \mathbb{G}_m$, which does not depend on the choice of isomorphism between $\mathrm{Aut}(U)$ and $\mathbb{G}_m$. This character is called the *weight* of U. The algebraic subgroups T and U generate an algebraic subgroup in $\mathrm{Aut}(X)$ isomorphic to $\mathbb{G}_a \rtimes_\chi T$.

Consider a nontrivial algebraic action of $\mathbb{G}_a$ on X, given by $\lambda \colon \mathbb{G}_a \to \mathrm{Aut}(X)$. If $f \in \mathcal{O}(X)$ is $\mathbb{G}_a$-invariant, then the *modification* $f \cdot \lambda$ of λ is defined in the following way:

$$(f \cdot \lambda)(r)x = \lambda(f(x)r)x$$

for $r \in \mathbb{C}$ and $x \in X$. This is again a $\mathbb{G}_a$-action. It is not difficult to see that if X is irreducible and $f \neq 0$, then $f \cdot \lambda$ and λ have the same invariants. If $U \subset \mathrm{Aut}(X)$ is a closed algebraic subgroup isomorphic to $\mathbb{G}_a$ and if $f \in \mathcal{O}(X)^U$ is a U-invariant, then in a similar way we define the modification $f \cdot U$ of U. Pick an isomorphism $\lambda \colon \mathbb{G}_a \to U$ and set

$$f \cdot U = \{(f \cdot \lambda)(r) \mid r \in \mathbb{G}_a\}.$$

2.4. **Divisible elements.** We call an element f in a group G *divisible by* n if there exists an element $g \in G$ such that $g^n = f$. An element is called *divisible* if it is divisible by all $n \in \mathbb{Z}^+$. If G is an agebraic group, then by [10, Lemma 3.12] for any $f \in G$ there exist $k > 0$ that depends on f such that f^k is a divisible element.

3. Proof of Theorem 1.1

The following lemma is well known and appeared in similar form in [4, Lemma 3.1].

Lemma 3.1. *Assume that* $\mathfrak{g}$ *is* $\mathbb{Z}^r$*-graded for* $r > 0$ *and consider a locally finite element* $z \in \mathfrak{g}$ *that does not belong to the zero component* $\mathfrak{g}_0$*. Then there exists a locally nilpotent homogeneous component of* z *of non-zero weight.*

Proof. Let us take the convex hull $P(z) \subset \mathbb{Z}^r \otimes \mathbb{Q}$ of component weights of z. Then for any non-zero vertex $v \in P(z)$ the corresponding homogeneous component is locally nilpotent. The details are left to the reader. □

Proof of Theorem 1.1. Assume first that $\mathrm{Aut}_{\mathrm{alg}}(X)$ contains a copy of $\mathbb{G}_a$ and $T = \mathbb{G}_m$. Let ∂ be a locally niplotent derivation corresponding to $\mathbb{G}_a$. By Lemma 3.1 there is a locally nilpotent derivation $\tilde{\partial} \in \mathrm{Der}\,\mathcal{O}(X)$ that is normalised by T. Denote by U a $\mathbb{G}_a$-action on X that corresponds to $\tilde{\partial}$. Hence, T acts on the ring of invariants $\mathcal{O}(X)^U$.

Claim 1. *There is a* T*-semi-invariant* $f \in \mathcal{O}(X)^U$ *of non-zero weight.*

Assume towards a contradiction that all invariants from $\mathcal{O}(X)^U$ are also T-invariants. This implies that

$$\mathcal{O}(X)^U = \mathcal{O}(X)^{T \ltimes U} = (\mathcal{O}(X)^U)^T. \tag{1}$$

This is possible only if $T \ltimes U$ acts with at most one-dimensional orbits since otherwise, if $T \ltimes U$ acts with a two-dimensional orbit, then by [5, Theorem 11.1.1.(7)] the quotient field of $\mathcal{O}(X)^{T \ltimes U}$ has transcendence degree $\dim X - 2$ which contradicts the fact that the transcendence degree of the quotient field of $\mathcal{O}(X)^U$ is $\dim X - 1$. Since $T \ltimes U$ is a connected algebraic group, these orbits are irreducible which are then isomorphic to $\mathbb{A}^1$ as U-orbits are closed subvarieties isomorphic to $\mathbb{A}^1$, see [5, Theorem 11.1.1]. We claim that $\mathcal{O}(X)^T = \mathcal{O}(X)^U$. Indeed, by (1) we have the inclusion $\mathcal{O}(X)^U \subset \mathcal{O}(X)^T$ and assume towards a contradiction that there is $f \in \mathcal{O}(X)^T \setminus \mathcal{O}(X)^U$. In another words, $f(t.x) = f(x)$ for all $x \in X$, $t \in T$, but there exists $x \in X$ such that $f(u.x) \neq f(x)$, where $u \in U$ is a non-trivial element. Hence, $U.x \simeq \mathbb{A}^1$. The torus T acts on $U.x$ since otherwise $T \ltimes U.x$ would be two-dimensional. Moreover, T acts on $U.x$ non-trivially since otherwise for any $y \in U.x$, $f(\overline{T.y}) = f(y)$ which implies that $f(tut^{-1}.y) = f(u.y)$. But this contradicts the fact that U acts on $U.x$ transitively since $tut^{-1} \neq u$. Hence, there is a quotient morphism

$$X \to X/\!\!/T = X/\!\!/U$$

Its fibers are at least one-dimensional and contain a unique closed T-orbit. Hence, fibers are one-dimensional and moreover, are isomorphic to U. By [9, Proposition 3.9.1] X is isomorphic to $\mathbb{A}^1 \times Y$, where Y is an affine variety isomorphic to $X/\!\!/T \simeq X/\!\!/U$ which contradicts the assumption on X.

Therefore, $\{f^k \cdot U \subset \mathrm{Aut}(X) \mid k \in \mathbb{N}\}$ are root subgroups with respect to T with different weights. Without loss of generality we can assume that U is a root subgroup with respect to T of non-zero weight since otherwise we can just replace U by $f \cdot U$.

Claim 2. *The subgroup*

$$G = T \ltimes (\bigoplus_{k\geq 1} f^k \cdot U) \subset \mathrm{Aut}(X)$$

is not linear.

Indeed, assume towards a contradiction that the subgroup $G \subset \mathrm{Aut}(X)$ is linear, i.e., there is an embedding $\varphi\colon G \to \mathrm{GL}_n(\mathbb{K})$. Since G is solvable, G can be embedded (after a necessarily conjugation) into a Borel subgroup $B \subset \mathrm{GL}_n(\mathbb{K})$ of upper triangular matrices. Moreover, the commutator $[G, G] = \bigoplus_{k\geq 1} f^k \cdot U$ embedds into $[B, B]$. In other words $\varphi(\bigoplus_{k\geq 1} f^k \cdot U)$ is a subgroup of the unipotent radical of B.

Consider the closed subgroup $\overline{\varphi(T)}^\circ \ltimes \overline{\varphi(f^k \cdot U)} \subset B \subset \mathrm{GL}_n(\mathbb{K})$. The subgroup $\overline{\varphi(f^k \cdot U)} = \overline{\varphi(f^k \cdot U)}^\circ \subset [B, B]$ is unipotent and $\overline{\varphi(T)} \subset B$ is an algebraic subgroup. Hence, $\overline{\varphi(T)}^\circ \subset \overline{\varphi(T)}$ is a finite index subgroup which implies that $\overline{\varphi(T)}^\circ$ contains infinitely many elements of finite order of $\varphi(T)$. As a consequence, $\overline{\varphi(T)}^\circ$ contains a copy of algebraic subtorus of positive dimension. Pick a big enough $k \in \mathbb{N}$ such that the kernel of T-action on $f^k \cdot U$ is $\langle \xi_k \rangle$, where ξ_k is an element of order bigger than the index $s = [\overline{\varphi(T)} : \overline{\varphi(T)}^\circ]$ and ξ_k acts on $\mathbb{K}f$ non-trivially. Hence, $\xi_k^s \in \overline{\varphi(T)}^\circ$ and since k is chosen to be big enough, we have that

$$\xi_k^s \text{ acts on } \mathbb{K}f \text{ non-trivially.} \tag{2}$$

Since $\varphi(\xi_k^s)$ centralizes $\varphi(f^k \cdot U)$, $\varphi(\xi_k^s)$ centralizes $\overline{\varphi(f^k \cdot U)}$ too. Choose a subtorus of $\overline{\varphi(T)}^\circ$ which we denote by $\tilde{T}$ that contains $\varphi(\xi_k^s)$. Pick $u_k \in \varphi(f^k \cdot U)$ and consider the unipotent subgroup $V_k = \langle \tilde{T}.u_k \rangle = \langle t u_k t^{-1} \mid t \in \tilde{T} \rangle \subset \mathrm{GL}_n(\mathbb{K})$. Note that $\tilde{T}$ normalizes V_k. Hence, the unipotet group V_k is a direct product of root subgroups with respect to $\tilde{T}$. The kernel of $\tilde{T}$-action on V_k contains $\langle \varphi(\xi_k^s) \rangle$. Since $\mathrm{GL}_n(\mathbb{K})$ is an algebraic group, i.e., is finitely dimensional, for a big enough k, the weights of all root subgroups of V_k with respect to $\tilde{T}$ are the same as the weights of the root subgroups of $V_{k+1} = \langle \tilde{T}.u_{k+1} \rangle$, where $u_{k+1} \in \varphi(f^{k+1} \cdot U) \subset [B, B] \subset \mathrm{GL}_n(\mathbb{K})$. Hence, $\langle \varphi(\xi_k^s) \rangle$ acts trivially on V_{k+1}. As a consequence, $\langle \xi_k^s \rangle$ acts trivially on $\varphi^{-1}(V_k) \subset f^k \cdot U$ and on $\varphi^{-1}(V_{k+1}) \subset f^{k+1} \cdot U$. Therefore, $\langle \xi_k^s \rangle$ acts trivially on $\overline{\varphi^{-1}(V_k)} \subset f^k \cdot U$ and on $\overline{\varphi^{-1}(V_{k+1})} \subset f^{k+1} \cdot U$ which implies that $\langle \xi_k^s \rangle$ acts trivially on $\mathbb{K}f$. This contradicts (2) which proves the theorem if X admits $\mathbb{G}_m$- and $\mathbb{G}_a$-actions.

If X admits two non-commuting $\mathbb{G}_m$-actions, then by Lemma 3.1 X admits a $\mathbb{G}_a$-action and the claim of the theorem follows from above. □

Remark 3.2. As it is already mentioned in the introduction, it is proved in [3] that $\mathrm{Aut}(\mathbb{A}^2)$ is not linear, i.e., it cannot be embedded into $\mathrm{GL}_n(\mathbb{K})$ for any $n \in \mathbb{N}$. This also follows from Theorem 1.1. Moreover, in [3, Proposition 2.3] it is proved that there is a countably generated subgroup of the subgroup

$$\mathrm{J} = \{(ax + c, by + f(x)) | a, b \in \mathbb{C}^*, c \in \mathbb{C}, f(y) \in \mathbb{C}[x]\} \subset \mathrm{Aut}(\mathbb{A}^2)$$

that is not linear. We note that by the Jung-Van der Kulk Theorem (see [7] and [8]) $\mathrm{Aut}(\mathbb{A}^2)$ is the amalgamated product of J and the group of affine transformations Aff_2 of $\mathbb{A}^2$ along their intersection C, i.e.,

$$\mathrm{Aut}(\mathbb{A}^2) = \mathrm{Aff}_2 *_C \mathrm{J}\,. \tag{3}$$

Using such an amalgamated product we claim that any representation of a subgroup

$$\mathrm{SAut}(\mathbb{A}^2)=\left\{(f,g)\in \mathrm{Aut}(\mathbb{A}^2)\mid \mathrm{jac}(f)=\det\begin{pmatrix}\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y}\\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}\end{pmatrix}=1\right\}$$

is trivial, i.e., any homomorphism $\varphi\colon \mathrm{SAut}(\mathbb{A}^2)\to \mathrm{GL}_n(\mathbb{K})$ is trivial. To show this we first note that the amalgamated product structure of $\mathrm{Aut}(\mathbb{A}^2)$ induces the amalgamated product structure of $\mathrm{SAut}(\mathbb{A}^2)$. More precisely, $\mathrm{SAut}(\mathbb{A}^2)$ is the amalgamated product of the group SAff_2 of special affine transformations of $\mathbb{A}^2$ and the subgroup

$$\mathrm{J}^s=\{(ax+c,by+f(x))|a,b\in\mathbb{C}^*,ab=1,c\in\mathbb{C},f(y)\in\mathbb{C}[x]\}\subset \mathrm{Aut}(\mathbb{A}^2).$$

By [3, Proposition 2.3] there is no embedding of J^s into $\mathrm{GL}_n(\mathbb{K})$. Hence, there is a non-identity element $g\in \mathrm{J}^s$ such that g is the kernel of φ. Therefore, the normal subgroup that contains g is also in the kernel of φ. But by [6] any normal subgroup that contains g coincides with $\mathrm{SAut}(\mathbb{A}^2)$ which proves the claim.

Moreover, any group homomorphism $\psi\colon \mathrm{Aut}(\mathbb{A}^2)\to \mathrm{GL}_n(\mathbb{K})$ factors through the homomorphism $\mathrm{jac}\colon \mathrm{Aut}(\mathbb{A}^2)\to \mathbb{G}_m$. Indeed, similarly as above, there is $g\in \mathrm{J}^s$ that is in the kernel of ψ. Hence, by the same argument as above the normal subgroup generated by g contains $\mathrm{SAut}(\mathbb{A}^2)$ and the claim follows.

4. Automorphism group that is isomorphic to a linear algebraic group

We begin this section with the lemma that is used in the proof of Theorem 4.2.

Lemma 4.1. *Let $\tilde{U},\tilde{V}\subset \mathrm{Aut}(\mathbb{A}^2)$ be two non-commuting unipotent subgroups. Then*

(1) the subgroup $G_{\tilde{U}}\subset \mathrm{Aut}(\mathbb{A}^2)$ generated by all one-dimensional unipotent subgroups that have the same generic orbits as $\tilde{U}$ coincides with its centralizer;

(2) the subgroup generated by $G_{\tilde{U}}$ and $G_{\tilde{V}}$ cannot be presented as a finite product of $G_{\tilde{U}}$ and $G_{\tilde{V}}$.

Proof. Recall that the group $\mathrm{Aut}(\mathbb{A}^2)$ has the amalgamated product structure $\mathrm{Aff}_2 *_C \mathrm{J}$, see (3). By [15] any closed algebraic subgroup of $\mathrm{Aut}(\mathbb{A}^2)$ is conjugate to one of the factors Aff_2 or J. Since $G_{\tilde{U}}\subset \mathrm{Aut}(\mathbb{A}^2)$ is infinite-dimensional, $G_{\tilde{U}}$ is conjugate to a subgroup of J. Moreover, since $G_{\tilde{U}}$ is infinite-dimensional, commutative and consists of unipotent elements, $G_{\tilde{U}}$ is conjugate to a subgroup of

$$\mathrm{J}_u=\{(x,y+f(x))\mid f(y)\in\mathbb{C}[x]\}\subset \mathrm{Aut}(\mathbb{A}^2).$$

Since $G_{\tilde{U}}$ is generated by all one-dimensional unipotent subgroups that have the same generic orbits, $G_{\tilde{U}}$ is conjugate to the whole J_u. It is easy to check that $\mathrm{J}_u\subset \mathrm{Aut}(\mathbb{A}^2)$ coincides with its centralizer which proves (1).

Without loss of generality we can assume that $G_{\tilde{U}}=\mathrm{J}_u$. Since $G_{\tilde{V}}$ does not commute with $G_{\tilde{U}}=\mathrm{J}_u$, $G_{\tilde{V}}$ is not a subgroup of J_u and since $G_{\tilde{V}}$ is infinite-diemnsional, $G_{\tilde{V}}$ is not a subgroup of J. Hence, (2) follows from amalgamated product structure of $\mathrm{Aut}(\mathbb{A}^2)$. □

Theorem 4.2. *Let X be an affine variety. If $\mathrm{Aut}(X)$ is isomorphic to a linear algebraic group as an abstract group, then the connected component $\mathrm{Aut}^\circ(X)$ is commutative. Moreover, in this case $\mathrm{Aut}^\circ(X)$ is either the algebraic torus or a direct limit of commutative unipotent groups.*

Proof of Theorem 4.2. Assume $\varphi\colon G\to \mathrm{Aut}(X)$ is an isomorphism of abstract groups, where G is a linear algebraic group. If the connected component $G^\circ\subset G$ is commutative, the finite index subgroup of $\mathrm{Aut}(X)$ is commutative. This implies that the connected component $\mathrm{Aut}^\circ(X)$ is commutative. Hence, by [2, Theorem B] (see also [13, Corollary

3.2]) $\mathrm{Aut}(X)^\circ$ is a union of commutative algebraic groups. The group $\mathrm{Aut}(X)^\circ$ either does not contain unipotent subgroups and in this case $\mathrm{Aut}^\circ(X)$ is isomorphic to an algebraic torus or $\mathrm{Aut}(X)^\circ$ contains unipotent subgroups. By Theorem 1.1 in the later case either $\mathrm{Aut}(X)$ does not contain a copy of $\mathbb{G}_m$ which implies that $\mathrm{Aut}^\circ(X)$ is the union of unipotent algebraic subgroups or X is isomorphic to $\mathbb{A}^1 \times Y$ for some affine variety Y. But $\mathrm{Aut}(\mathbb{A}^1 \times Y)$ is non-commutative. This proves the theorem in case G° is commutative.

We assume now that G° is non-commutative. In this case G contains closed connected commutative subgroups U and V that do not commute and U normalizes V. Indeed, if G is non-unipotent, it contains a maximal subtorus $T \subset G$ and a root subgroup normalized but not centralized by T. If G is unipotent, then G is nilpotent, i.e.,

$$G = G_0 \rhd G_1 \rhd \cdots \rhd G_n = \{\mathrm{id}\}, \tag{4}$$

where $G_{i+1} = [G, G_i]$, $[G, G_i] = \{ghg^{-1}h^{-1} \mid g \in G, h \in G_i\}$. In particular, G_{n-1} is a subgroup of the center of G. Moreover, for any $H \subset G_{n-2} \setminus G_{n-1}$ isomorphic to $\mathbb{G}_a$, the group $V = H \times G_{n-1}$ is commutative. Choose a subgroup $U \subset G \setminus V$ isomorphic to $\mathbb{G}_a$ that does not commute with V. Note that such U exists as G is non-commutative. Moreover, we claim that U normalizes V. Indded, $[U, V] \subset [G, G_{n-2}] = G_{n-1}$ which means that $uvu^{-1}v^{-1} \in G_{n-1}$ for any $u \in U$, $v \in V$. Hence, $uvu^{-1} \in G_{n-1}v \subset V$ which proves the claim.

Hence, $\overline{\varphi(U)}^\circ, \overline{\varphi(V)}^\circ \subset \mathrm{Aut}(X)$ are closed connected commutative subgroups. Since $\varphi(U)$ normalizes $\varphi(V)$, $\varphi(U)$ normalizes $\overline{\varphi(V)} \subset \mathrm{Aut}(X)$ and hence $\overline{\varphi(U)}^\circ$ normalizes $\overline{\varphi(V)}^\circ$. By [2, Theorem B] (see also [13, Corollary 3.2]) $\overline{\varphi(U)}^\circ, \overline{\varphi(V)}^\circ \subset \mathrm{Aut}(X)$ are unions of algebraic subgroups and hence $\overline{\varphi(U)}^\circ \ltimes \overline{\varphi(V)}^\circ$ is a union of algebraic subgroups. Note that $\overline{\varphi(U)}^\circ$ and $\overline{\varphi(V)}^\circ$ do not commute since otherwise the subgroups $\varphi^{-1}(\overline{\varphi(U)}^\circ)$ and $\varphi^{-1}(\overline{\varphi(V)}^\circ)$ of G would commute which is not the case as $\varphi^{-1}(\overline{\varphi(U)}^\circ) \cap U \subset U$ is a dense subgroup and analogously $\varphi^{-1}(\overline{\varphi(U)}^\circ) \cap U \subset U$ is a dense subgroup. Therefore, there are non-commuting algebraic subgroups in $\mathrm{Aut}(X)$.

Claim 3. $\mathrm{Aut}_{\mathrm{alg}}(X) \subset \mathrm{Aut}(X)$ *does not contain a copy of* $\mathbb{G}_m$.

Since $\mathrm{Aut}_{\mathrm{alg}}(X)$ is not commutative, by Theorem 1.1 $\mathrm{Aut}_{\mathrm{alg}}(X)$ can contain a copy of $\mathbb{G}_m$ only if X is isomorphic to a product $\mathbb{A}^1 \times Y$, where Y is an affine variety that does not admit a regular action of a positive-dimensional algebraic group. So, assume $X \simeq \mathbb{A}^1 \times Y$ and $Y \subset \mathbb{A}^l$ is a closed subset. The automorphism group $\mathrm{Aut}(\mathbb{A}^1 \times Y)$ has the following form:

$$\{(x, y_1, \ldots, y_l) \mapsto (g(y)x + h(y), h_1(x, y), \ldots, h_l(x, y)) \mid y = (y_1, \ldots, y_l) \in Y, \tag{5}$$

$$g(y) \in \mathcal{O}(Y)^*, h \in \mathcal{O}(Y), h_i \in \mathcal{O}(\mathbb{A}^1 \times Y)\}, \tag{6}$$

where $i = 1, \ldots, l$ and the map $Y \to Y$, $(y_1, \ldots, y_l) \mapsto (h_1(x, y), \ldots, h_l(x, y))$ is an automorphism for each $x \in \mathbb{A}^1$. Consider the one-dimensional algebraic subtorus

$$T = \{(x, y_1, \ldots, y_l) \mapsto (ax, y_1, \ldots, y_l) \mid \ a \in \mathbb{K}^*\} \subset \mathrm{Aut}(\mathbb{A}^1 \times Y)$$

and the subgroup

$$\mathrm{Aut}^u(\mathbb{A}^1 \times Y) = \{(x, y_1, \ldots, y_l) \mapsto (x + h(y), y_1, \ldots, y_l) \mid \ h \in \mathcal{O}(Y)\} \subset \mathrm{Aut}(\mathbb{A}^1 \times Y).$$

Any automorphism of $\mathrm{Aut}(\mathbb{A}^1 \times Y)$ that commutes with each element of T has the form

$$(x, y_1, \ldots, y_l) \mapsto (g(y)x, h_1(x, y), \ldots, h_l(x, y)), \tag{7}$$

where $g \in \mathcal{O}(Y)^*$ and h_i does not depend on the first coordinate. This follows from the following equality

$$(g(y)x, h_1(x,y), \dots, h_l(x,y)) = (a^{-1}x, y_1, \dots, y_l) \circ (g(y)x, h_1(x,y), \dots, h_l(x,y)) \circ (ax, y_1, \dots, y_l) = (g(y)x, h_1(ax,y), \dots, h_l(ax,y)).$$

Now, since T acts on $\mathrm{Aut}^u(\mathbb{A}^1 \times Y)$ by conjugations, $\overline{\varphi^{-1}(T)}$ acts on $\overline{\varphi^{-1}(\mathrm{Aut}^u(\mathbb{A}^1 \times Y))}$ by conjugations. Note that $\overline{\varphi^{-1}(T)}$ contains an algebraic subtorus $\tilde{T}$ of positive dimension as $\varphi^{-1}(T)$ contains infinitely many elements of finite order. The image of $\tilde{T}$ does not contain elements of the form

$$(x, y_1, \dots, y_l) \mapsto (g(y)x, y_1, \dots, y_l),$$

where $g(y) \in \mathcal{O}(Y)^* \setminus \mathbb{K}$ because such elements are not divisible by high enough positive integer and all elements of the algebraic torus are divisible (see Section 2.4).

The subgroup $\tilde{T} \subset \overline{\varphi^{-1}(T)}$ acts on $\overline{\varphi^{-1}(\mathrm{Aut}^u(\mathbb{A}^1 \times Y))}$ by conjugations and for each

$$u_n = \{(x, y_1, \dots, y_l) \mapsto (x + h_n(y), y_1, \dots, y_l) \mid h_n \in \mathcal{O}(Y)\} \subset \mathrm{Aut}(\mathbb{A}^1 \times Y),$$

where $\{h_n\}$ form a basis of the vector space $\mathcal{O}(Y)$, the subgroup $\langle \tilde{T}.\varphi^{-1}(u_n)\rangle = \langle t\varphi^{-1}(u_n)t^{-1} \mid t \in \tilde{T}\rangle$ is an algebraic subgroup of G as the orbit $\tilde{T}.\varphi^{-1}(u_n) = \{t\varphi^{-1}(u_n)t^{-1} \mid t \in \tilde{T}\} \subset G$ is constructible. Moreover, by (5) and because each element of $\tilde{T}$ has the form

$$(x, y_1, \dots, y_l) \mapsto (ax, h_1(y), \dots, h_l(y)), \tag{8}$$

the subgroup generated by $\langle \tilde{T}.\varphi^{-1}(u_n)\rangle$ is a subgroup of $\varphi^{-1}(u_n \cdot \mathrm{Tr})$, where

$$\mathrm{Tr} = \{(x, y_1, \dots, y_l) \mapsto (x + b, y_1, \dots, y_l) \mid b \in \mathbb{K}\} \subset \mathrm{Aut}(\mathbb{A}^1 \times Y).$$

Hence, for any natural k the subgroup generated by $\langle \tilde{T}.\varphi^{-1}(u_n)\rangle, \dots, \langle \tilde{T}.\varphi^{-1}(u_{n+k})\rangle$ is an algebraic subgroup of dimension at least k. But this is not possible as G is an algebraic group, i.e., is finite-dimensional. We arrive to the contadiction which proves the claim.

Hence, $\overline{\varphi(U)}^\circ \ltimes \overline{\varphi(V)}^\circ$ is the union of unipotent subgroups and in particular it contains a non-commutative unipotent subgroup W that acts on X with a two-dimensional orbit O that is isomorphic to $\mathbb{A}^2$, see [5, Theorem 11.1.1]. Pick subgroups $\tilde{U} \subset \overline{\varphi(U)}^\circ$ and $\tilde{V} \subset \overline{\varphi(V)}^\circ$ that do not commute, are isomorphic to $\mathbb{G}_a$ and generate the subgroup that acts with a two-dimensional orbit $O \simeq \mathbb{A}^2$. Take the maximal commutative subgroups H_1 and H_2 of $\mathrm{Aut}(X)$ that contain $\mathcal{O}(X)^{\tilde{U}} \cdot \tilde{U}$ and $\mathcal{O}(X)^{\tilde{V}} \cdot \tilde{V} \subset \mathrm{Aut}(X)$ respectively. Therefore, $\varphi^{-1}(H_1), \varphi^{-1}(H_2) \subset G$ are closed subgroups. Hence, the subgroup $H \subset G$ generated by $\varphi^{-1}(H_1)$ and $\varphi^{-1}(H_2)$ can be presented as a finite product of subgroups $\varphi^{-1}(H_1)$ and $\varphi^{-1}(H_2)$. On the other hand, the subgroup of $\mathrm{Aut}(X)$ generated by H_1 and H_2 cannot be presented as a finite product of subgroups H_1 and H_2. Indeed, by Lemma 4.1(1) the restriction of H_1 to $O \simeq \mathbb{A}^2$ is the subgroup of $\mathrm{Aut}(O \simeq \mathbb{A}^2)$ generated by all one-dimensional unipotent subgroups that have the same generic orbits as $\tilde{U}\mid_O$. Analogous situation we have with H_2. Now by Lemma 4.1(2), the subgroup $H = \langle H_1, H_2\rangle \subset \mathrm{Aut}(X)$ restricted to $O \simeq \mathbb{A}^2$ cannot be presented as a finite product of H_1 and H_2 restricted to O. We arrive to the contradiction and finish the proof. □

5. Proof of Theorem 1.2

We start this section with the next proposition.

Proposition 5.1. *Assume $\mathbb{K}$ is uncountable and X is a connected affine variety. Then* $\mathrm{Aut}(X)$ *is not isomorphic to the Cremona group* $\mathrm{Bir}(\mathbb{A}^n) = \mathrm{Bir}(\mathbb{P}^n)$ *as an abstract group for any $n > 0$.*

Proof. The proof of this statement is similar to the proof of Theorem A in [2]. We give some details here for the convenience of the reader. Let

$$\mathrm{Tr} = \{(x_1, \ldots, x_n) \mapsto (x_1 + c_1, \ldots, x_n + c_n) \mid c_i \in \mathbb{K}\} \subset \mathrm{Aut}(\mathbb{A}^n) \subset \mathrm{Bir}(\mathbb{A}^n)$$

be the subgroup of all translations and Tr_i be the subgroup of translations of the i-th coordinate:

$$(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_i + c, \ldots, x_n), \tag{9}$$

where c in $\mathbb{K}$. Let $\mathrm{T} \subset \mathrm{GL}_n(\mathbb{K}) \subset \mathrm{Aut}(\mathbb{A}^n) \subset \mathrm{Bir}(\mathbb{P}^n)$ be the diagonal group (viewed as a maximal torus) and let T_i be the subgroup of automorphisms

$$(x_1, \ldots, x_n) \mapsto (x_1, \ldots, ax_i, \ldots, x_n), \tag{10}$$

where $a \in \mathbb{K}^*$. A direct computation shows that Tr *(resp. T) coincides with its centralizer in* $\mathrm{Bir}(\mathbb{A}^n) = \mathrm{Bir}(\mathbb{P}^n)$. Assume towards a contradiction that there is an isomorphism $\varphi\colon \mathrm{Bir}(\mathbb{A}^n) \to \mathrm{Aut}(X)$ of abstract groups. Similarly as in [2, Lemma 5.2] the groups $\varphi(\mathrm{Tr})$, $\varphi(\mathrm{Tr}_i)$, $\varphi(T)$ and $\varphi(T_i)$ are closed subgroups of $\mathrm{Aut}(X)$ for all $i = 1, \ldots, n$. Now the proof of [2, Theorem A] implies that $X \simeq \mathbb{A}^n$ and $\varphi(T) \subset \mathrm{Aut}(X \simeq \mathbb{A}^n)$ is isomorphic to the n-dimensional algebraic torus. Assume $U \subset \mathrm{PGL}_{n+1}(\mathbb{K}) \subset \mathrm{Bir}(\mathbb{A}^n)$ is a root subgroup with respect to T. This means that T acts on U with two orbits. Therefore, $\varphi(U) \subset \mathrm{Aut}(X)$ is a constructible subset which is a group. We conclude that $\varphi(U) \subset \mathrm{Aut}(X)$ is an algebraic subgroup. Moreover, since $\mathrm{PGL}_{n+1}(\mathbb{K})$ is generated by its finitely many root subgroups U with respect to T, $\varphi(\mathrm{PGL}_{n+1}(\mathbb{K}))$ is generated by finitely many algebraic subgroups $\varphi(U)$ which implies that $\varphi(\mathrm{PGL}_{n+1}(\mathbb{K})) \subset \mathrm{Aut}(X \simeq \mathbb{A}^n)$ is an algebraic subgroup. Further, $\varphi(T) \subset \varphi(\mathrm{PGL}_{n+1}(\mathbb{K}))$ is a maximal subtorus that is isomorphic to $\mathbb{G}_m^n$ which means that $\varphi(\mathrm{PGL}_{n+1}(\mathbb{K}))$ is a simple algebraic group of rank n that is isomorphic to $\mathrm{PGL}_{n+1}(\mathbb{K})$ as an abstract group. We conclude that $\varphi(\mathrm{PGL}_{n+1}(\mathbb{K}))$ is isomorphic to $\mathrm{PGL}_{n+1}(\mathbb{K})$ as an algebraic group. But this is not possible since the algebraic group $\mathrm{PGL}_{n+1}(\mathbb{K})$ does not act regularly on $\mathbb{A}^n$ as the only closed subgroup H of codimension $\leq n$ of $\mathrm{PGL}_{n+1}(\mathbb{K})$ is a maximal parabolic subgroup such that $\mathrm{PGL}_{n+1}(\mathbb{K})/H \simeq \mathbb{P}^n$. We arrive to a contradiction with the isomorphism $X \simeq \mathbb{A}^n$. The proof follows. □

Proof of Theorem 1.2. Let $H \subset \mathrm{Bir}(X)$ be a maximal algebraic subtorus. By [12, Theorem 1] X is birationally equivalent to $\mathbb{A}^l \times Z$, where H acts on $\mathbb{A}^l$ with an open orbit and $Z \subset \mathbb{A}^r$ is an affine variety with a trivial action of H. By Proposition 5.1 we can assume that Z is positive dimensional. Assume there is an isomorphism $\varphi\colon \mathrm{Bir}(X) = \mathrm{Bir}(\mathbb{A}^l \times Z) \to \mathrm{Aut}(Y)$. Consider the maximal commutative subgroup G of $\mathrm{Bir}(\mathbb{A}^l \times Z)$ of the form

$$\{(x_1, \ldots, x_l, z_1, \ldots, z_r) \mapsto (f_1(z)x_1, \ldots, f_l(z)x_l, g_1(z), \ldots, g_r(z)) \mid f_i(z), g_i(z) \in \mathbb{K}(Z)\}$$

that contains the commutative subgroup

$$\{(x_1, \ldots, x_l, z_1, \ldots, z_r) \mapsto (f_1(z)x_1, \ldots, f_l(z)x_l, z_1, \ldots, z_r) \mid f_i(z) \in \mathbb{K}(Z)\},$$

where the map

$$Z \to Z \ \ (z_1, \ldots, z_r) \mapsto (g_1(z), \ldots, g_r(z))$$

is a birational transformation of Z.

Claim 4. *The subgroup $G \subset \mathrm{Bir}(\mathbb{A}^l \times Z)$ coincides with its centralizer.*

To prove this claim consider a birational transformation ϕ of $\mathbb{A}^l \times Z$ of the form

$$(x_1, \ldots, x_l, z_1, \ldots, z_r) \mapsto (F_1(x,z), \ldots, F_l(x,z), G_1(x,z), \ldots, G_r(x,z)),$$

where $F_i(x,z), G_i(x,z) \in \mathbb{K}(\mathbb{A}^l \times Z)$, $x = (x_1, \ldots, x_l)$, $z = (z_1, \ldots, z_r)$ that commutes with each element from G. Hence, ϕ commutes with $T \subset G$, i.e., with all birational transformations of $\mathbb{A}^l \times Z$ of the form

$$(x_1, \ldots, x_l, z_1, \ldots, z_r) \mapsto (t_1x_1, \ldots, t_lx_l, z_1, \ldots, z_r), \;\; t_1, \ldots, t_r \in \mathbb{K}^*.$$

Direct computations show that

$$t_iF_i(t_1^{-1}x_1, \ldots, t_l^{-1}x_l, z_1, \ldots, z_r) = F_i(x_1, \ldots, x_l, z_1, \ldots, z_r)$$

and

$$G_j(t_1^{-1}x_1, \ldots, t_l^{-1}x_l, z_1, \ldots, z_r) = G_j(x_1, \ldots, x_l, z_1, \ldots, z_r)$$

for all $t_1, \ldots, t_r \in \mathbb{K}^*$, $i = 1 \ldots, l$, $j = 1, \ldots, r$. Therefore, $F_i(x,z) = h_i(z)x_i$ for some $h_i(z) \in \mathbb{K}(Z)$ and $G_j \in \mathbb{K}(Z)$. This proves the claim.

By [10, Lemma 2.4] $\varphi(G) \subset \operatorname{Aut}(Y)$ is a closed ind-subgroup and by [2, Theorem B] (see also [13, Corollary 3.2]) the connected component $\varphi(G)^\circ \subset \operatorname{Aut}(Y)$ is the union of commutative algebraic subgroups. Since $\varphi(G)^\circ \subset \varphi(G)$ is a countable index subgroup, there is an element $g = (f_1(z)x_1, x_2, \ldots, x_l, z_1, \ldots, z_r) \in G$ with non-constant f_1 such that $\varphi(g)$ belongs to $\varphi(G)^\circ$. Since $\varphi(G)^\circ$ is the union of connected algebraic groups, $\varphi(g)$ belongs to a connected algebraic subgroup of $\varphi(G)^\circ$ and hence there exists $k > 0$ such that $\varphi(g)^k$ is a divisible element (see Section 2.4). The element g^k again has a form $(\tilde{f}_1(z)x_1, x_2, \ldots, x_l, z_1, \ldots, z_r)$ with a non-constant $\tilde{f}_1 = \frac{r_1}{r_2}$, where $r_1, r_2 \in \mathcal{O}(Z)$. Moreover, $g^k \in G$ is not divisible. More precisely, without loss of generality we can assume that r_1 is non-constant and hence, there is no $h \in G$ such that $h^{\deg r_1 + 1} = g^k$. Indeed, otherwise there would exist a rational function $s \in \mathbb{K}(Z)$ such that $s^{\deg r_1 + 1} = \tilde{f}_1 = \frac{r_1}{r_2}$ which is not the case. We get the contradiction which proves the theorem. □

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