AUTOMORPHISM GROUPS OF AFFINE VARIETIES WITHOUT NON-ALGEBRAIC ELEMENTS

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Abstract. Given an affine algebraic variety X, we prove that if the neutral component $\mathrm{Aut}^\circ(X)$ of the automorphism group consists of algebraic elements, then it is nested, i.e., is a direct limit of algebraic subgroups. This improves our earlier result [5]. To prove it, we obtain the following fact. If a connected ind-group G contains a closed connected ind-subgroup $H \subset G$ with a geometrically smooth point, and for any $g \in G$ some power of g belongs to H, then $G = H$.

1. Introduction

In this note we work over an algebraically closed field of characteristic zero $\mathbb{K}$. We study the automorphism groups of affine varieties. It is well known that these groups can be larger than any algebraic group. For example, the automorphism group $\mathrm{Aut}(\mathbb{A}^n)$ of the affine n-space $\mathbb{A}^n$ contains a copy of a polynomial ring in $n-1$ variables, hence it is infinite-dimensional for $n \geq 2$.

In [9] Shafarevich introduced the notion of the infinite-dimensional algebraic group, which is currently called the *ind-group* and showed that $\mathrm{Aut}(\mathbb{A}^n)$ has the structure of the ind-group. Later it was shown that $\mathrm{Aut}(X)$ has a natural structure of an ind-group for any affine variety X, see [4, Section 5] and also [6, Section 2].

We call an element g of the automorphism group $\mathrm{Aut}(X)$ *algebraic* if there is an algebraic subgroup G of the ind-group $\mathrm{Aut}(X)$ that contains g. We also denote by $\mathbb{G}_a$ the additive group of the field and by $\mathcal{U}(X) \subset \mathrm{Aut}(X)$ the (possibly trivial) subgroup generated by all the $\mathbb{G}_a$-actions. It is usually called the *special automorphism group* and is also denoted by $\mathrm{SAut}(X)$.

In [5] we proved that for the subgroup $\mathrm{Aut}_{\mathrm{alg}}(X) \subset \mathrm{Aut}(X)$ generated by all connected algebraic subgroups the following conditions are equivalent:

- $\mathcal{U}(X)$ is abelian;
- all elements of $\mathrm{Aut}_{\mathrm{alg}}(X)$ are algebraic;
- the subgroup $\mathrm{Aut}_{\mathrm{alg}}(X) \subset \mathrm{Aut}(X)$ is a closed nested ind-subgroup, i.e., is a direct limit of algebraic subgroups;
- $\mathrm{Aut}_{\mathrm{alg}}(X) = \mathbb{T} \ltimes \mathcal{U}(X)$, where $\mathbb{T}$ is a maximal subtorus of $\mathrm{Aut}(X)$, and $\mathcal{U}(X)$ is closed in $\mathrm{Aut}(X)$.

In this paper we prove that this result can be partially extended from $\mathrm{Aut}_{\mathrm{alg}}(X)$ to the connected component $\mathrm{Aut}^\circ(X)$. More precisely, we have the following result which is proved in Section 4.

Theorem 1.1. *Let X be an affine variety. The following conditions are equivalent:*

(1) all elements of $\mathrm{Aut}^\circ(X)$ are algebraic;

(2) the subgroup $\mathrm{Aut}^\circ(X) \subset \mathrm{Aut}(X)$ is a closed nested ind-subgroup;

[a]The research of the first author was carried out at the HSE University at the expense of the Russian Science Foundation (project no. 21-71-00062).

(3) $\mathrm{Aut}^\circ(X) = \mathbb{T} \ltimes \mathcal{U}(X)$*, where* $\mathbb{T}$ *is a maximal subtorus of* $\mathrm{Aut}(X)$*, and* $\mathcal{U}(X)$ *is abelian and consists of all unipotent elements of* $\mathrm{Aut}(X)$*.*

In [6] this theorem is proved for algebraic surfaces with a nontrivial group $\mathcal{U}(X)$.

The key observation in our proof is as follows. Under condition (1) for any element of $\mathrm{Aut}^\circ(X)$ some power of it belongs to $\mathbb{T} \ltimes \mathcal{U}(X)$, see Lemma 4.1. In Section 3 we prove that an ind-group G coincides with its ind-subgroup H if for any element of G some power of it lies in H, see Theorem 3.1. We also need a certain smoothness condition on H, which is fulfilled if H is nested.

In Section 5 we also state some observations about the group of automorphisms of a rigid affine variety, i.e., an affine variety that admits no $\mathbb{G}_a$-actions.

2. Preliminaries

2.1. **Ind-groups.** The notion of an ind-group goes back to Shafarevich who called these objects infinite dimensional groups (see [9]). We refer to [4] for basic notions in this context.

Definition 2.1. By an affine *ind-variety* we mean an injective limit $V = \varinjlim V_i$ of an ascending sequence $V_0 \hookrightarrow V_1 \hookrightarrow V_2 \hookrightarrow \ldots$ such that the following holds:

(1) $V = \bigcup_{k\in\mathbb{N}} V_k$;
(2) each V_k is an affine algebraic variety;
(3) for all $k \in \mathbb{N}$ the embedding $V_k \hookrightarrow V_{k+1}$ is closed in the Zariski topology.

For simplicity we will call an affine ind-variety simply an ind-variety. An ind-variety V has a natural *topology*: a subset $S \subset V$ is called open (resp. closed) if $S_k := S \cap V_k \subset V_k$ is open (resp. closed) for all $k \in \mathbb{N}$. A closed subset $S \subset V$ has a natural structure of an ind-variety and is called an ind-subvariety.

The product of ind-varieties $X = \varinjlim X_i$ and $Y = \varinjlim Y_i$ is defined as $\varinjlim X_i \times Y_i$. A *morphism* between ind-varieties $V = \bigcup_k V_k$ and $W = \bigcup_m W_m$ is a map $\phi : V \to W$ such that for every $k \in \mathbb{N}$ there is an $m \in \mathbb{N}$ such that $\phi(V_k) \subset W_m$ and that the induced map $V_k \to W_m$ is a morphism of algebraic varieties. This allows us to give the following definition.

Definition 2.2. An ind-variety G is said to be an *ind-group* if the underlying set G is a group such that the map $G \times G \to G$, $(g, h) \mapsto gh^{-1}$, is a morphism.

A *closed subgroup* H of G is a subgroup that is also a closed subset. Then H is again an ind-group with respect to the induced ind-variety structure. A closed subgroup H of an ind-group $G = \varinjlim G_i$ is called an *algebraic subgroup* if H is contained in some G_i.

The next result can be found in [4, Section 5].

Proposition 2.3. *Let X be an affine variety. Then* $\mathrm{Aut}(X)$ *has the structure of an ind-group such that a regular action of an algebraic group G on X induces a homomorphism of ind-groups* $G \to \mathrm{Aut}(X)$*.*

Two ind-structures $V = \varinjlim V_i$ and $V = \varinjlim V_i'$ are called *equivalent*, if the identity map $\varinjlim V_i \to \varinjlim V_i'$ is an isomorphism of ind-varieties. One also calls $\varinjlim V_i'$ an *admissible* filtration of the ind-variety $V = \varinjlim V_i$.

Definition 2.4 ([4, Definition 1.9.4])**.** A point p in an ind-variety V is called *geometrically smooth*, if there exists an admissible filtration $V = \varinjlim V_i$ such that p is a smooth point of V_i for each i.

An element $g \in \operatorname{Aut}(X)$ is called *algebraic* if there is an algebraic subgroup $G \subset \operatorname{Aut}(X)$ such that $g \in G$. An ind-group $G = \varinjlim G_i$ is called *nested* if G_i is an algebraic group for $i = 1, 2, \ldots$.

2.2. **Lie algebras of ind-groups.** For an ind-variety $V = \bigcup_{k \in \mathbb{N}} V_k$ we can define the tangent space in $x \in V$ in the obvious way: we have $x \in V_k$ for $k \geq k_0$, and $T_x V_k \subset T_x V_{k+1}$ for $k \geq k_0$, and then we define

$$T_x V := \bigcup_{k \geq k_0} T_x V_k,$$

which is a vector space of at most countable dimension.

For an ind-group G, the tangent space $T_e G$ has a natural structure of a Lie algebra which is denoted by $\operatorname{Lie} G$, see [7, Section 4] and [4, Section 2] for details.

2.3. $\mathbb{G}_a$**-actions.** Given an affine variety X, we denote by $\operatorname{Aut}_{\mathrm{alg}}(X) \subset \operatorname{Aut}(X)$ the subgroup generated by all connected algebraic subgroups of the automorphism group $\operatorname{Aut}(X)$.

An element $u \in \operatorname{Aut}(X)$ is called *unipotent* if u belongs to an algebraic subgroup of $\operatorname{Aut}(X)$ isomorphic to $\mathbb{G}_a$. We denote the automorphism subgroup of $\operatorname{Aut}(X)$ generated by all the unipotent elements by $\mathcal{U}(X)$.

3. Ind-subgroup with powers of elements

In this section we explore the situation when an ind-subgroup contains some power of any element of the ind-group and prove Theorem 3.1.

Theorem 3.1. *Let G be a connected ind-group and $H \subset G$ be a closed connected ind-subgroup with a geometrically smooth point. Assume that for any $g \in G$ there exists $d \in \mathbb{N}$ such that $g^d \in H$. Then $G = H$.*

By [4, Theorem 0.1.1] and [4, Remark 2.2.3] there exist ind-structures $G = \varinjlim G_i$ and $H = \varinjlim H_i$ such that each G_i and H_i is an irreducible subset containing the identity. Moreover, since there exists a geometrically smooth point $p \in H$, then every point in H is geometrically smooth, and we may assume that each H_i is smooth at the identity.

Remark 3.2. Any nested ind-group is geometrically smooth at each point. However, to our knowledge, this property is not proven for arbitrary ind-groups. For example, a stronger property of being *strongly smooth* does not hold for the ind-group $\operatorname{Aut}(\mathbb{A}^2)$, see [4, Corollary 14.1.2]. More generally, this group does not admit a filtration by normal varieties.

Consider the multiplication map

$$\mu_d \colon G^d = \underbrace{G \times \cdots \times G}_{d \text{ times}} \to G, \ (g_1, \ldots, g_d) \mapsto g_1 \cdots g_d.$$

Its differential is the linear map

$$\mathrm{d}\mu_d \colon (\operatorname{Lie} G)^d = \underbrace{\operatorname{Lie} G \times \cdots \times \operatorname{Lie} G}_{d \text{ times}} \to \operatorname{Lie} G.$$

We have the following statement.

Lemma 3.3. *Given $(x_1, \ldots, x_d) \in (\operatorname{Lie} G)^d$, the following holds:*

$$\mathrm{d}\mu_d((x_1, \ldots, x_d)) = x_1 + \cdots + x_d.$$

Proof. By linearity,

$$\mathrm{d}\mu_d((x_1,\dots,x_d)) = \sum_i \mathrm{d}\mu_d((0,\dots,0,x_i,0,\dots,0)). \tag{1}$$

We claim that $\mathrm{d}\mu_d((0,\dots,0,x_i,0,\dots,0)) = x_i$. Indeed, let us denote

$$s_i \colon G \to \underbrace{G\times\cdots\times G}_{d \text{ times}},\ g \mapsto (id,\dots,\underbrace{g}_{i\text{-th position}},\dots,id).$$

The composition $\mu_d \circ s_i$ is the trivial automorphism of G. Hence,

$$\mathrm{d}(\mu_d\circ s_i)\colon\ \operatorname{Lie} G \xrightarrow{\mathrm{d}s_i} \underbrace{\operatorname{Lie} G\oplus\cdots\oplus\operatorname{Lie} G}_{d \text{ times}} \xrightarrow{\mathrm{d}\mu_d} \operatorname{Lie} G \tag{2}$$

is the identity map, where the first map in (2) is given by the embedding into the i-th coordinate. Therefore, we conclude that $\mathrm{d}\mu_d((0,\dots,0,x_i,0,\dots,0)) = x_i$. Now, from (1) it follows that

$$\mathrm{d}\mu_d((x_1,\dots,x_d)) = \sum_i x_i.$$

□

Definition 3.4. We denote $\phi_d\colon G \to G,\ g\mapsto g^d$. It is an endomorphism of an ind-variety.

Corollary 3.5. *The differential* $\mathrm{d}\phi_d\colon\ \operatorname{Lie} G \to \operatorname{Lie} G$ *satisfies*

$$\mathrm{d}\phi_d(x) = d\cdot x$$

for any $x \in \operatorname{Lie} G$.

Proof. Consider an embedding

$$s\colon G \to \underbrace{G\times\cdots\times G}_{d \text{ times}};\quad g\mapsto (g,\dots,g).$$

Its differential is the embedding

$$\mathrm{d}s\colon\ \operatorname{Lie} G \to \underbrace{\operatorname{Lie} G\oplus\cdots\oplus\operatorname{Lie} G}_{d \text{ times}};\ \ x\mapsto (x,\dots,x).$$

Since $\phi_d = \mu_d\circ s$, by Lemma 3.3

$$\mathrm{d}\phi_d(x) = \mathrm{d}\mu_d((x,\dots,x)) = d\cdot x. \tag{3}$$

□

Definition 3.6. For each $d,k\in\mathbb{N}$ we denote

$$X_{d,k} = \phi_d^{-1}(H_k) = \{g\in G \mid g^d\in H_k\}\subset G.$$

Lemma 3.7. *(1) The subset $X_{d,k}$ is closed in G for any $d,k\in\mathbb{N}$.*
(2) For any closed algebraic subset $A\subset G$ there exist $d,k\in\mathbb{N}$ such that $A\subset X_{d,k}$.

Proof. The map ϕ_d is a morphism of ind-varieties, so the first statement follows from $X_{d,k} = \phi_d^{-1}(H_k)$.

The increasing sequence of closed subsets

$$X_{1!,1}\subset X_{2!,2}\subset\ldots\subset X_{i!,i}\subset\ldots$$

exhausts G, hence $A\subset X_{i!,i}$ for some $i\in\mathbb{N}$. We may take $d=i!$ and $k=i$ to get the second assertion. See also [4, Theorem 1.3.3]. □

Proof of Theorem 3.1. Denote the restriction of $\phi_d \colon G \to G,\ g \mapsto g^d$, to $X_{d,k}$ by $\phi_{d,k}$. Then

$$\phi_{d,k} \colon X_{d,k} \to H_k, \ \ g \mapsto g^d.$$

Its differential map at the identity,

$$d(\phi_{d,k})_{\mathrm{id}} \colon T_{\mathrm{id}} X_{d,k} \to T_{\mathrm{id}} H_k,$$

is given by $x \mapsto d \cdot x$ due to Corollary 3.5. This map has trivial kernel and is surjective due to $H_k \subset X_{d,k}$. So, $\dim T_{\mathrm{id}} X_{d,k} = \dim T_{\mathrm{id}} H_k$.

Since H_k is smooth at the identity, $\dim T_{\mathrm{id}} H_k = \dim H_k$. Let Y be the union of irreducible components of $X_{d,k}$ containing the identity. From $H_k \subset Y$ and $\dim T_{\mathrm{id}} Y = \dim H_k$ we infer that $Y = H_k$. Thus, the set $X_{d,k}$ contains H_k as an irreducible component, and other components do not contain the identity.

By Lemma 3.7, for any $i \in \mathbb{N}$ there exist $d, k \in \mathbb{N}$ such that $G_i \subset X_{d,k}$. Since G_i is irreducible and contains the identity, G_i is a subset of the only irreducible component of $X_{d,k}$ which contains the identity, namely, H_k. We conclude that $G \subseteq H$. ☐

4. Neutral component without non-algebraic elements

In this section we assume that $\mathrm{Aut}^\circ(X)$ consists of algebraic elements. By [5, Main Theorem], $\mathcal{U}(X)$ is an abelian unipotent ind-group (which is trivial, one-dimensional, or infinite-dimensional), and the subgroup $\mathrm{Aut}_{\mathrm{alg}}(X)$ generated by connected algebraic subgroups equals $\mathbb{T} \ltimes \mathcal{U}(X)$, where $\mathbb{T}$ is a maximal algebraic torus.

Lemma 4.1. *For any algebraic element $g \in \mathrm{Aut}^\circ(X)$ there exists $d \in \mathbb{N}$ such that $g^d \in \mathbb{T} \ltimes \mathcal{U}(X)$.*

Proof. The Zariski closure of $\{g^n \mid n \in \mathbb{Z}\}$ is an abelian algebraic group, which we denote by G. The subgroup G° is of finite index in G, so we may denote $d = |G/G^\circ|$ and we have $g^d \in G^\circ$. Since G° is a connected algebraic group, $G^\circ \subset \mathbb{T} \ltimes \mathcal{U}(X)$. The claim follows. ☐

Remark 4.2. By [2, Theorem 1.1], for any algebraic group G there is a finite subgroup $H \subset G$ such that $G = H \cdot G^\circ$. Thus, any algebraic element of $\mathrm{Aut}(X)$ is a product of an element of $\mathrm{Aut}_{\mathrm{alg}}(X)$ and a finite order one.

As we have mentioned above, $\mathcal{U}(X)$ is a direct limit of its unipotent algebraic subgroups, i.e., $\mathcal{U}(X) = \varinjlim \mathcal{U}(X)_k$, where each $\mathcal{U}(X)_k$ is a closed unipotent algebraic subgroup of $\mathcal{U}(X)$. We set $\mathcal{U}(X)_k = \mathcal{U}(X)$ for each k if $\mathcal{U}(X)$ is itself an algebraic group.

Proof of Theorem 1.1. Assume that all elements of $\mathrm{Aut}^\circ(X)$ are algebraic. By [5, Theorem 1.3], $\mathrm{Aut}_{\mathrm{alg}}(X)$ equals $\mathbb{T} \ltimes \mathcal{U}(X)$. By Lemma 4.1, we may apply Theorem 3.1 to $G = \mathrm{Aut}^\circ(X)$ and $H = \mathbb{T} \ltimes \mathcal{U}(X)$ and conclude that $\mathrm{Aut}^\circ(X) = \mathbb{T} \ltimes \mathcal{U}(X)$. This proves the implication (1) ⇒ (3). The implications (3) ⇒ (2) ⇒ (1) are obvious. ☐

Corollary 4.3. *Let X be an affine algebraic variety without $\mathbb{G}_a$-actions such that $\mathrm{Aut}^\circ(X)$ consists of algebraic elements. Then $\mathrm{Aut}^\circ(X)$ is an algebraic torus of dimension at most $\dim X$.*

Proof. In this case $\mathrm{Lie}\,\mathrm{Aut}^\circ(X) = \mathfrak{t}$, so $\mathrm{Aut}^\circ(X)$ is finite-dimensional. Then $\mathrm{Aut}^\circ(X)$ is a connected algebraic group and is defined by $\mathrm{Lie}\,\mathbb{T}$. ☐

Remark 4.4. If X does not admit $\mathbb{G}_a$- and $\mathbb{G}_m$-actions, then Theorem 1.1 can be obtained from [1, Proposition 3.6]. Indeed, in this case all elements of $\mathrm{Aut}^\circ(X)$ are of finite order. Hence, for some $n \in \mathbb{N}$ the subset of elements of order at most n, which is a closed, contains the identity as a limit point.

5. The automorphism group of a rigid variety

In this section we do not assume that $\mathrm{Aut}^\circ(X)$ consists of algebraic elements. Assume that an affine variety X is rigid, i.e., admits no $\mathbb{G}_a$-actions. By [5, Main Theorem], all $\mathbb{G}_m$-actions on X commute, hence $\mathrm{Aut}_{\mathrm{alg}}(X) = \mathbb{T}$ is an algebraic torus.

Proposition 5.1. *Each element of* $\mathrm{Aut}^\circ(X)$ *commutes with* $\mathbb{T}$.

Proof. The torus $\mathbb{T}$ is a normal closed subgroup in $\mathrm{Aut}^\circ(X)$. Consider the action of $\mathrm{Aut}^\circ(X)$ on $\mathbb{T}$ by conjugations. Since the group of automorphisms of the algebraic torus $\mathbb{T}$ of dimension n seen as an algebraic group is isomorphic to $\mathrm{GL}(n, \mathbb{Z})$, we obtain the homomorphism $\mathrm{Aut}^\circ(X) \to \mathrm{GL}(n, \mathbb{Z})$. Since $\mathrm{GL}(n, \mathbb{Z})$ is discrete, the image of $\mathrm{Aut}^\circ(X)$ is trivial. The assertion follows. □

Corollary 5.2. *Each element of* $\mathrm{Aut}^\circ(X)$ *is contained in an abelian group* $A \times \mathbb{T}$*, where* A *is a cyclic group.*

Remark 5.3. Any maximal abstract abelian subgroup G of $\mathrm{Aut}^\circ(X)$ is an at most countable extension of $\mathbb{T}$. Indeed, G coincides with its centralizer, hence is a closed ind-subgroup ([8, Lemma 2.4]). Further, G contains $\mathbb{T}$, and by [3, Theorem B] the connected component G° is algebraic. So, $G^\circ = \mathbb{T}$.

In particular, the only maximal connected abelian ind-subgroup of $\mathrm{Aut}^\circ(X)$ is $\mathbb{T}$.

Question 5.4. Given a rigid affine variety X, what can we say about the subset of algebraic elements of $\mathrm{Aut}^\circ(X)$?

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