On the automorphism group of an affine variety

Andriy Regeta

University of Jena

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Similarly as the group of linear automorphisms of the affine n-space $\mathbb{A}^n$ which is usually denoted by GL_n plays an important role in the theory of algebraic groups, the group of regular automorphisms of $\mathbb{A}^n$ should play an important role in the study of the infinite-dimensional algebraic groups which are usually called ind-groups.

Ind-groups

Ind-structure on $\mathrm{Aut}(\mathbb{A}^n)$.

Any automorphism $f \in \mathrm{Aut}(\mathbb{A}^n)$ is given by $(f_1, \dots, f_n)$, where $f_i \in \mathbb{K}[x_1, \dots, x_n]$. Define

$$\mathrm{Aut}(\mathbb{A}^n)_d = \{f = (f_1, \dots, f_n) \in \mathrm{Aut}(\mathbb{A}^n) \mid \deg f = \max_i \deg f_i, \deg f^{-1} \le d\}$$

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$S \subset \mathrm{Aut}(\mathbb{A}^n)$ is called closed (open) if $S \cap \mathrm{Aut}(\mathbb{A}^n)_d \subset \mathrm{Aut}(\mathbb{A}^n)_d$ is closed (open) for each $d \in \mathbb{N}$.

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The map $\varphi\colon \mathrm{Aut}(\mathbb{A}^n) \times \mathrm{Aut}(\mathbb{A}^n) \to \mathrm{Aut}(\mathbb{A}^n)$, $(g, h) \mapsto gh^{-1}$ is the morphism of ind-varieties.

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The group $\mathrm{Aut}(\mathbb{A}^2)$ is already much bigger. In particular, $\mathrm{Aut}(\mathbb{A}^2)$

- contains the subgroup $\{(x, y) \mapsto (x, y + f(x)) \mid f \in \mathbb{C}[x]\}$
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- We know essentially nothing about the closed connected subgroups of $\mathrm{Aut}(\mathbb{A}^n)$.

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Perepechko-R.

Let X be an affine variety. The following conditions are equivalent:

1. all elements of $\mathrm{Aut}^\circ(X)$ are algebraic;
2. the subgroup $\mathrm{Aut}^\circ(X) \subset \mathrm{Aut}(X)$ is a closed nested ind-subgroup;
3. $\mathrm{Aut}^\circ(X) = T \ltimes U(X)$, where T is a maximal subtorus of $\mathrm{Aut}(X)$, and $U(X)$ is abelian and consists of all unipotent elements of $\mathrm{Aut}(X)$.

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Example

Blanc and Dubouloz constracted a family of rational affine surfaces S with huge groups of automorphisms in the following sense: the normal subgroup $\mathrm{Aut}(S)_{\mathrm{alg}}$ of $\mathrm{Aut}(S)$ generated by all algebraic subgroups of $\mathrm{Aut}(S)$ is not generated by any countable family of such subgroups, and the quotient $\mathrm{Aut}(S)/\mathrm{Aut}(S)_{\mathrm{alg}}$ contains a free group over an uncountable set of generators.

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So, we know essentially nothing about the automorphism group of an affine surface.

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Question (propaganded in particular by Schafarevich and Serre)

Which properties of GL_n extend to $\mathrm{Aut}(X)$ and $\mathrm{Bir}(X)$?

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3. All Borel subgroups of a linear algebraic group are conjugate.
An analogous statement does not hold for ind-groups. For example, not all maximal solvable subgroups of $\mathrm{Aut}(\mathbb{A}^3)$ are conjugate.

However, there is a chance to prove that a maximal connected solvable subgroup of $\mathrm{Aut}(\mathbb{A}^n)$ has solvability length $\leq n+1$ and such subgroups of solvability length $n+1$ are conjugate.

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4. Assume G and H are reductive algebraic groups and $\varphi\colon G \to H$ is an isomorphism of abstract groups. Then $G \simeq H$ as an algebraic group.

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4. Assume G and H are reductive algebraic groups and $\varphi \colon G \to H$ is an isomorphism of abstract groups. Then $G \simeq H$ as an algebraic group. Analogous result does not hold for ind-groups. Define

$$\mathrm{D}_p = \{xy = p(z)\} \subset \mathbb{A}^3.$$

For two generic D_p and D_q the is an isomorphism of abstract groups $\mathrm{Aut}(\mathrm{D}_p) \simeq \mathrm{Aut}(\mathrm{D}_q)$, but these groups are not isomorphic as ind-groups (Leuenberger-R.).

Theorem A (Cantat-Xie-R.)

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Corollary (Cantat-Xie-R.)

A commutative connected closed subgroup of $\mathrm{Aut}(X)$ is the union of linear algebraic groups.

Consider a nontrivial $\mathbb{G}_a$-action on X, given by $\lambda\colon \mathbb{G}_a \to \mathrm{Aut}(X)$. If $f \in \mathcal{O}(X)$ is $\mathbb{G}_a$-invariant, then the *modification* $f \cdot \lambda$ of λ is defined in the following way:

$$(f \cdot \lambda)(s)x = \lambda(f(x)s)x$$

for $s \in \mathbb{C}$ and $x \in X$. It is again a $\mathbb{G}_a$-action. If X is irreducible and $f \neq 0$, then $f \cdot \lambda$ and λ have the same invariants.

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If $U \subset \mathrm{Aut}(X)$ is a closed subgroup isomorphic to $\mathbb{G}_a$ and if $f \in \mathcal{O}(X)^U$ is a U-invariant, then in a similar way we define the modification $f \cdot U$ of U. Choose an isomorphism $\lambda\colon \mathbb{C}^+ \to U$ and set

$$f \cdot U = \{(f \cdot \lambda)(s) \mid s \in \mathbb{G}_a\}.$$

Application of Theorem A

A maximal commutative connected closed subgroup of $\mathrm{Aut}(X)$ that does not contain semisimple elements equals

$$\left\{ f \cdot u \in Q(\mathcal{O}(X)^U) \cdot U \mid f \cdot u \text{ is an automorphism of } X \right\}$$

where $u \in U$ and U is an algebraic unipotent subgroup of $\mathrm{Aut}(X)$. Moreover, this subgroup is a maximal commutative subgroup of $\mathrm{Aut}(X)$.

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where $u \in U$ and U is an algebraic unipotent subgroup of Aut(X). Moreover, this subgroup is a maximal commutative subgroup of Aut(X). As a consequence, any isomorphism $\varphi \colon \mathrm{Aut}(X) \to \mathrm{Aut}(Y)$ sends a unipotent element to a unipotent element.

Theorem A does not hold for the group of birational transformations

For $X = \mathbb{A}^2$ consider

$$V = \{(x, y) \mapsto (x, (1 + ax)y) \mid a \in \mathbb{C}\}.$$

Then $\mathrm{id} \in V$, but $\langle V \rangle$ is not an algebraic group since it is not of bounded degree.

Theorem B (van Santen-R.)

Let X be a G-spherical affine variety different from algebraic torus and Y be an affine irreducible normal variety. If there is an isomorphism $\varphi\colon \mathrm{Aut}(Y) \simeq \mathrm{Aut}(X)$, then $\varphi(G) \subset \mathrm{Aut}(Y)$ is a reductive subgroup, Y is $\varphi(G)$-spherical and $\Lambda^+(X) = \Lambda^+(Y)$.

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Example (the case of algebraic torus)

Let T be an algebraic torus and let C be a smooth affine curve. If C has trivial automorphism group and no invertible global functions, then there is an isomorphism $\mathrm{Aut}(T) \to \mathrm{Aut}(C \times T)$ that preserves algebraic subgroups.

Projective case

Note that there is no projective variety determined by its automorphism group in a reasonably big category of algebraic varieties.

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Example

Most toric projective varieties have automorphism group isomorphic to algebraic torus. Hence, projective toric variety is not determined by its automorphism group.

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Theorem(Cantat, Xie)

Let X be an n-dimensional quasi-projective variety, where $n \geq 2$. If $\mathrm{Bir}(X)$ is isomorphic to $\mathrm{Bir}(\mathbb{P}^n)$, then X is rational.

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Moreover, if X is a variety of dimension n and there exist an injective morphism of groups $\mathrm{SL}(n+1, \mathbb{Z}) \hookrightarrow \mathrm{Bir}(X)$, then X is rational.

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Task

Show that $\mathbb{P}^n$ is uniquely determined (up to birational equivalence) among all irreducible varieties by its group of birational transformations.

Idea of the proof of Theorem 1

- (1) $V \subset \mathrm{Aut}(X)$ is an irreducible algebraic subvariety;
- (2) Define $W = VV^{-1}$. It is constructible subset of $\mathrm{Aut}(X)$; We have a sequence of constructible subsets $W \subset W^2 \subset W^3 \subset \dots$ and to prove the theorem it is enough to show that $W^i = W^{i+1}$ for some $i \in \mathbb{N}$;
- (3) There is a Zariski dense open subset U of X and an integer k_0 such that $\dim(W^k(x)) = s_X$ for all $k \geq k_0$ and all $x \in U$.
- (4) There is an integer $l \geq 0$ such that for every $x \in X$, $W^l(x) = \langle W \rangle(x)$ and $W^l(x)$ is an open subset of $\overline{\langle W \rangle(x)}$.

$\langle W \rangle$ acts with a dence orbit

If $\langle W \rangle$ acts on X with an orbit of dimension $\dim X$, then $\langle W \rangle$ acts on X with an open orbit $\langle W \rangle(x_0) = W^I(x_0)$. Hence, for any $f \in \langle W \rangle$ there exists $g \in W^I$ such that $f(x_0) = g(x_0)$ or equivalently $fg^{-1}(x_0) = x_0$. Moreover, since $\langle W \rangle$ is commutative, $h(x_0) = hfg^{-1}(x_0) = fg^{-1}(h(x_0))$ for any $h \in \langle W \rangle$. Since $\langle W \rangle$ acts on X with an open orbit, $f = g$ and so $\langle W \rangle = W^I$.

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- By (3) thetre is an integer $l > 0$ and a W-invariant open subset $U \subset X$ such that $s(x) = s_X$ and $W^l(x) = \langle W \rangle(x)$ for every $x \in U$.
- One of the crucial steps is to prove an analog of Rosenlicht's quotient theorem for $\langle W \rangle$: we show that there is an open subset $Y \subset X$ such that the geometric quotient $Y/\langle W \rangle$ exists and we can construct (after possible shrinking of $Y/\langle W \rangle$ and Y respectively) a dominant morphism $\pi \colon Y \to B$ such that:
  1. very fiber of π, in particular its generic fiber, is geometrically irreducible;
  2. the generic fiber of π is normal and affine, shrinking B (and Y accordingly) again, we may assume B and Y to be normal and affine;
  3. the action of $\langle W \rangle$ on the generic fiber Y_η has bounded degree.

Denote by $\mathbb{A}_B^N$ the affine N-space over $\mathcal{O}(B)$ and by $\mathrm{Aut}_B(\mathbb{A}_B^n)$ the group of automorphisms $g \in \mathrm{Aut}(\mathbb{A}_B^n)$ such that $\psi = \psi \circ g$, where $\psi \colon \mathbb{A}_B^n \to B$ is the projection morphism.

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Black box

1. There exist an embedding $\tau \colon Y \to \mathbb{A}_B^N$ for some $N \geq 0$ and a homomorphism $\rho \colon \langle W \rangle \to \mathrm{GL}_N(\mathcal{O}(B)) \subset \mathrm{Aut}_B(\mathbb{A}_B^N)$ such that
$$\tau \circ g = \rho(g) \circ \tau \qquad (\forall g \in \langle W \rangle).$$
2. The image of ρ is a subgroup of $U_B(B) \times \mathbb{G}_{m,B}^s(B) \subset \mathrm{GL}_N(\mathcal{O}(B))$.
3. The ind-groups $U_B(B)$ and $U_B(B)$ are increasing unions of algebraic subgroups.
4. We conclude that the image of ρ is contained in the algebraic subgroup of $U_B(B) \times \mathbb{G}_{m,B}^s(B)$ and so the image of ρ is an algebraic subgroup of $\mathrm{GL}_N(\mathcal{O}(B)) \subset \mathrm{Aut}_B(\mathbb{A}_B^N)$.

Recall Theorem B

Definition

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Weight Monoid

$\mathcal{O}(X)$ is a multiplicity free G-module, that is, the multiplicity of every irreducible module in $\mathcal{O}(X)$ is at most 1. By the weight monoid $\Lambda^+(X)$ of X we mean the set of all highest weights of the G-module $\mathcal{O}(X)$.

Recall Theorem B

Theorem B (van Santen-R.)

Let X be a G-spherical affine variety different from algebraic torus and Y be an affine irreducible normal variety. If there is an isomorphism $\varphi\colon \mathrm{Aut}(Y) \simeq \mathrm{Aut}(X)$, then $\varphi(G) \subset \mathrm{Aut}(Y)$ is a reductive subgroup, Y is $\varphi(G)$-spherical and $\Lambda^+(X) = \Lambda^+(Y)$.

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Theorem C (van Santen, R.)

Let T be an algebraic torus and let Y be an irreducible normal affine variety such that $\mathrm{Aut}(T)$, $\mathrm{Aut}(Y)$ are isomorphic and $\dim Y \leq \dim T$. Then T, Y are isomorphic as varieties.

Corollary

- if X is toric, then $Y \simeq X$.
- if X and Y are smooth, then $Y \simeq X$.
- in general, for a given X there are finitely many spherical varieties $Y_1, \ldots, Y_l$ such that $\mathrm{Aut}(Y_j) \simeq \mathrm{Aut}(X)$.

Remark

Note that if Y is quasi-projective in the Theorem above, then the result does not hold: for example,

$$\mathrm{Aut}(X) \simeq \mathrm{Aut}(X \times Z),$$

where Z is projective with trivial automorphism group.
Moreover, the condition on Y to be irreducible is crucial:

$$\mathrm{Aut}(X) \simeq \mathrm{Aut}(X \sqcup Z),$$

where Z is an affine variety with the trivial automorphism group.

What if Y is non-normal?

Conjecture/Proposition(Diaz, Liendo, R.)

An affine toric variety X different from algebraic torus is determined by its automorphism groups in the category of all affine irreducible varieties if and only if $X \simeq \mathbb{A}^1 \times Z$ for some toric Z.

The main issue

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There exists an affine surface and an isomorphism $\psi : \mathrm{Aut}(S) \xrightarrow{\sim} \mathrm{Aut}(S)$ such that for any algebraic subgroup $H \subset \mathrm{Aut}(S)$, $\psi(H)$ is not an algebraic subgroup of $\mathrm{Aut}(S)$.

Sketch of the proof

$\varphi\colon \mathrm{Aut}(X) \to \mathrm{Aut}(Y)$ is an isomorphism. The image of G is an algebraic subgroup of $\mathrm{Aut}(Y)$ isomorphic to G as an algebraic group.

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Definition

Let X be an affine spherical G-variety. A unipotent subgroup $H \subset \mathrm{Aut}(X)$ is called a generalized root subgroup (with respect to B) if H is commutative and every one-dimensional subgroup of H is normalized by B.

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Definition

Let X be an affine spherical G-variety. A unipotent subgroup $H \subset \mathrm{Aut}(X)$ is called a generalized root subgroup (with respect to B) if H is commutative and every one-dimensional subgroup of H is normalized by B. the weights of all the one-dimensional subgroups of a generalized root subgroups are the same. This weight we call the weight of the generalized root group.

φ sends generalized root subgroup $\mathrm{Aut}(X)$ with respect to $B \subset G$ to a generalized root subgroup of $\mathrm{Aut}(Y)$ with respect to $\varphi(B) \subset \varphi(G)$ of the same dimension with the same weight.

Sketch of the proof

Proposition

Let Y be an irreducible normal affine G-variety. The following statements are equivalent:

- Y is G-spherical;
- there exists a constant C such that $\dim H \leq C$ for each generalized root subgroup $H \subset \mathrm{Aut}(Y)$.

Since φ sends generalized root subgroups to generalized root subgroups of the same dimension, Y is $\varphi(G)$-spherical.

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We study a so-called asymptotic cones of weights of root subgroups. Since φ sends root subgroups to root subgroups of the same weight (this is difficult), we conclude that $\Lambda^+(X) = \Lambda^+(Y)$.

For the simplicity we concentrate only on toric case

A maximal subtorus $G = T \subset \mathrm{Aut}(X)$ of dimension $\dim X$ coincides with its centralizer. Hence, $\varphi(T) \subset \mathrm{Aut}(Y)$ also coincides with its centralizer which implies that $\varphi(T) \subset \mathrm{Aut}(Y)$ is closed.

But we do not know a priori if $\varphi(T) \subset \mathrm{Aut}(Y)$ is connected!

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A subgroup $\varphi(T)^\circ \subset \mathrm{Aut}(Y)$ is a commutative connected non-trivial subgroup. Hence, by Theorem A it is a union of algebraic subgroups of $\mathrm{Aut}(Y)$. Moreover, $\varphi(T)^\circ$ does not contain unipotent elements, since otherwise, if there is a unipoptent element $u \in \varphi(T)^\circ$, then $\varphi^{-1}(u) \in T$ is a unipotent element (by Application of Theorem A) which is not the case. Therefore, $\varphi(T)^\circ$ is isomorphic to an algebraic torus.

There is a trick how to show that $\varphi(T)^\circ$ is actually $\dim X = n$-dimensional.

Root subgroups are sent to root subgroups

All the generalized root subgroups of $\mathrm{Aut}(X)$ with respect to $G = T$ are one-dimensional subgroups $U \simeq \mathbb{G}_a$ and have non-trivial different weights. Moreover, T acts on U with two orbits.

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Since each unipotent element is sent to a unipotent element, $\varphi(U) \subset \mathrm{Aut}(Y)$ consists of unipotent elements. Therefore, $\overline{\varphi(U)} \subset \mathrm{Aut}(Y)$ is connected group which is a direct limit of unipotent algebraic groups.

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Since $\varphi(T)^\circ \subset \varphi(T)$ is a subgroup of countable index, $\varphi(T)$ acts on $\varphi(U)$ with countably many orbits

$$\dim \varphi(U) \leq \dim \varphi(T)^\circ = \dim T.$$

Root subgroups are sent to root subgroups

Moreover, since T acts on U with the kernel that contains a subgroup isomorphic to $(\mathbb{Z}/p\mathbb{Z})^{n-1}$ for any $p \in \mathbb{N}$, $\varphi(T)$ also acts on $\varphi(U)$ with the kernel that contains a subgroup isomorphic to $(\mathbb{Z}/p\mathbb{Z})^{n-1}$. Therefore, $U \simeq \mathbb{G}_a$.

Why $\varphi(T)^\circ = \varphi(T)$?

One can choose a product of n commuting root subgroups $V \subset \mathrm{Aut}(X)$ with respect to T. T acts on V with a finite kernel. Hence, $\varphi(V)$ is also a product of n commuting root subgroups with respect to $\varphi(T)^\circ$. This implies that both $\varphi(T)^\circ$ act on $\varphi(V)$ with a finite kernel.

Therefore, $\varphi(T)^\circ \subset \varphi(T)$ is a subgroup of finite index.

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Therefore, $\varphi(T)^\circ \subset \varphi(T)$ is a subgroup of finite index.

Since both groups $\varphi(T)^\circ$ and $\varphi(T)$ are isomorphic to $\mathbb{G}_m^n$ as abstract groups, all elements of these groups are divisible, i.e., one can take a root of any order of each element inside of the group.

Since any element in a finite group is not divisible, we conclude that $\varphi(T)^\circ = \varphi(T)$.

$\dim Y = \dim X$.

There a subtorus $D \subset T \subset \mathrm{Aut}(X)$ of dimension $n-1$ such that all root subgroups of $\mathrm{Aut}(X)$ with respect to D coincide with the root subgroups of $\mathrm{Aut}(X)$ with respect to T. In particular, all root subgroups of $\mathrm{Aut}(X)$ with respect to D have different weights.

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The difficulty now is to prove that $\varphi(D) \subset \mathrm{Aut}(Y)$ is a closed algebraic subtorus of dimension $n-1$ and all root subgroups of $\mathrm{Aut}(Y)$ with respect to $\varphi(D)$ have different weights. This implies that $\mathcal{O}(Y)^{\varphi(U)}$ is multiplicity free $\varphi(T)$-module and hence $\dim Y \leq \dim \varphi(D) + 1 = n$ which implies that Y is toric.

End of the proof

Why $U \subset \mathrm{Aut}(X)$ and $\varphi(U) \subset \mathrm{Aut}(Y)$ have the same weight?

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Hence, the set of weights of root subgroups of $\mathrm{Aut}(X)$ with respect to T coincides with the set of weights of root subgroups of $\mathrm{Aut}(Y)$ with respect to $\varphi(T)$.
Combinatorial work shows that the polyhedral cones that determine X and Y are the same. Therefore, $Y \simeq X$.

Further work

1. Prove some analog (a weaker version) of Theorem A for the group of birational transformations and show that a rational variety is detemined (up to birational equivalence) by its group of birational transformations;
2. Characterize Borel subgroups of $\mathrm{Aut}(X)$ and of $\mathrm{Bir}(X)$. Find out more structural results for $\mathrm{Aut}(X)$ and for $\mathrm{Bir}(X)$.

THANK YOU FOR YOUR ATTENTION