Groups of automorphisms of some affine varieties

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Andriy Regeta
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auf Antrag von

Prof. Dr. Hanspeter Kraft

Dr. Jean-Philippe Furter

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Prof. Dr. Jörg Schibler, Dekan

To my parents.

Contents.

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2. Introduction

In 1872 Felix Klein published his inauguration paper named *Vergleichende Betrachtungen ueber neuere geometrische Forschungen* (see [Kle93]) for his professorship at the University of Erlangen (Bavaria, Germany). This paper acquired world-wide fame among mathematicians under the name of *Erlangen Programm*. Klein proposed that group theory, a branch of mathematics that uses algebraic methods to abstract the idea of symmetry, was the most useful way of organizing geometrical knowledge. One can translate it into the modern mathematical language as follows.

Study of geometrical objects via their transformation (automorphism, birational transformation etc.) groups.

This approach was very fruitful in many areas of mathematics, for example, to study manifolds via their diffeomorphism group, field extensions via their Galois group, algebraic varieties via their automorphisms.

In particular, Richard P. Filipkewicz proved that a real connected manifold is determined by its group of diffeomorphism i.e. if M and N are real connected manifolds of class C^k and C^j respectively, then an isomorphism $\phi : \mathrm{Diff}^k(M) \to \mathrm{Diff}^j(N)$ of abstract groups implies equality $j = k$ and that there exists a diffeomorphism $\psi : M \to N$ of class C^k.

In this thesis, we focus on the study of affine varieties via their automorphisms. Shafarevich introduced the structure of an infinite dimensional variety on the automorphism group $\mathrm{Aut}(\mathbb{A}^n)$, a so-called ind-variety (see Section 3). The aim of this thesis is: Study the automorphism group of affine varieties within the framework of ind-varieties.

The thesis is organized as follows. In Section 3 we introduce the basic concepts and notions that we will need. In Section 4 we give an overview of the results in the articles of this thesis. Thereafter we list all these articles. We work over the field of complex numbers $\mathbb{C}$ (but all results hold true over algebraically closed field of characteristic zero) if not explicitely stated otherwise.

3. Fundamentals

3.1. **Ind-groups and their Lie algebras.** In [Sha66] Shafarevich introduced the notion of an *infinite dimensional algebraic group* or shortly *ind-group* (see also [Kum02]). It was introduced in order to study the automorphism group $\mathrm{Aut}(\mathbb{A}^n)$ of the affine n-space. Recently, Furter-Kraft and Dubouloz independently showed that $\mathrm{Aut}(X)$ has the structure of an ind-group for any affine variety X.

Definition 1. An *ind-variety* is a set X together with a filtration $X_1 \subset X_2 \subset ...$ with the following properties

(a) $X = \bigcup_{i=1}^{\infty} X_i$;

(b) each X_n has the structure of an algebraic variety;

(c) the inclusion $X_n \subset X_{n+1}$ is a closed immersion.

In this case we denote $X = \varinjlim X_i$. In case each X_i is affine we call X an affine ind-variety. We endow each ind-variety $X = \varinjlim X_i$ with the following so-called *ind-topology*: a subset $A \subset X$ is called closed (resp. open) if and only if $A \cap X_i$ is closed (resp. open) in X_i for all i.

Example 1. (1) Any (finite-dimensional) variety X is of course canonically an ind-variety, where we take each $X_n = X$.

(2) If X and Y are ind-varieties, then $X \times Y$ is canonically an ind-variety, where we define the filtration by

$$(X \times Y)_n = X_n \times Y_n,$$

and we put the product variety structure on $X_n \times Y_n$.

(3) $\mathbb{A}^\infty = \{(a_1, a_2, a_3, ...) : a_i \in \mathbb{C}$ and all but finitely many a_i's are zero$\}$ is an ind-variety under the filtration: $\mathbb{A}^1 \subset \mathbb{A}^2 \subset \mathbb{A}^3 \subset ...$, where $\mathbb{A}^n \subset \mathbb{A}^\infty$ is the set of all the sequences with $a_{n+1} = a_{n+2} = ... = 0$, which can be identified with the n-dimensional affine space.

(4) Any countable infinite set $S = \{x_0, x_1, ...\}$ is an ind-variety under the filtration $S_n = \{x_0, x_1, ..., x_n\}$ which has the structure of a variety.

(5) Any vector space V of countable dimension over the field $\mathbb{C}$ is an affine ind-variety. Take a basis $\{e_i\}_{i \geq 1}$ of V. This gives rise to a $\mathbb{C}$-linear isomorphism $\mathbb{A}^\infty \mapsto \sum a_i e_i$. By transporting the ind-variety structure from $\mathbb{A}^\infty$ via this isomorphism, we get an (affine) ind-variety structure on V. It is easy to see that a different choice of basis of V gives an equivalent ind-variety structure on V.

Definition 2. A *morphism* between ind-varieties $V = \bigcup_k V_k$ and $W = \bigcup_m W_m$ is a map $\phi : V \to W$ such that for any k there is an m such that $\phi(V_k) \subset W_m$ and the induced map $V_k \to W_m$ is a morphism of varieties. *Isomorphism* of ind-varieties as well as products of ind-varieties are defined in the usual way. This allows to define an *ind-group* as an ind-variety G with a group structure such that multiplication $G \times G \to G : (g, h) \mapsto g \cdot h$, and taking the inverse $G \to G : g \to g^{-1}$, are both morphisms.

In a similar way we define the notion of ind-semigroup. Similarly as $\mathrm{Aut}(X)$ is an ind-group, the semigroup of endomorphisms $\mathrm{End}(X)$ and the semigroup of dominant maps $\mathrm{Dom}(X)$ have the structures of an ind-semigroups for any affine variety X.

Definition 3. For any affine ind-variety $X = \varinjlim X_i$, the morphisms $X \to \mathbb{A}^1$ are the elements of $\varprojlim \mathcal{O}(X_i)$. We call these morphisms the *regular functions* on X and we denote $\mathcal{O}(X) := \varprojlim \mathcal{O}(X_i)$.

A closed subgroup of an ind-group $G = \varinjlim G_i$ is called *algebraic* if it is contained in some G_i. We call an element $g \in G$ *algebraic* if the closure of the group generated by g is an algebraic subgroup of G.

Definition 4. A map $f : X \to Y$ of ind-varieties is called a *closed embedding*, or equivalently, a *closed immersion*, if for any n there exists $m(n)$ such that $f(X_n) \subset Y_{m(n)}$ and $f|_{X_n} : X_n \to Y_{m(n)}$ is a closed embedding of varieties, $f(X)$ is closed in Y and moreover, $f : X_n \to f(X_n)$ is an isomorphism of varieties.

An ind-variety X is called *irreducible* if the underlying topological space is irreducible, i.e. X is not the union of two proper closed subsets. Similarly, X is called *connected* if the underlying topological space is connected.

Definition 5. Let X be an ind-variety with filtration (X_n). For any $x \in X$, define the *Zariski tangent space* $T_x(X)$ of X at x by

$$T_x(X) = \varinjlim T_x(X_n),$$

where $T_x(X_n)$ is the Zariski tangent space of X_n at x. Note that $x \in X_n$ for all large enough n.

A morphism $f : X \to Y$ induces a linear map $(df)_x : T_x(X) \to T_{f(x)}(Y)$, for any $x \in X$, called the *differential of f at x*. Moreover, this satisfies the chain rule: $(d(g \circ f))_x = (dg)_{f(x)} \circ (df)_x$, for a morphism $g : Y \to Z$. Hence, an isomorphism $f : X \to Y$ of ind-varieties induces an isomorphism $(df)_x : T_x(X) \to T_{f(x)}(Y)$, for any $x \in X$.

Proposition 1. [Kum02, Proposition 4.2.2] *For an ind-group H, the Zariski space $T_e(H)$ at the identity element e is endowed with a natural structure of a Lie algebra which will be denoted by* $\mathrm{Lie}\, H$*.*

Moreover, if $f : G \to H$ is a group morphism between ind-groups, then the derivative $(df)_e : \mathrm{Lie}\, G \to \mathrm{Lie}\, H$ is a Lie algebra homomorphism.

As in the case of algebraic groups, an ind-group G is connected if and only if G is irreducible (see [Kum02, Lemma 4.2.5]).

Definition 6. Let G be an ind-group. An algebraic element $g \in G$ is called unipotent if it is either trivial or if the closure of the group $\langle g \rangle$ generated by g is isomorphic to the additive group $G_a := \mathbb{C}^+$. The subset of all unipotent elements of G is denoted by G_u.

3.2. **Automorphisms of affine varieties.** By an *endomorphism* of the affine n-space $\mathbb{A}^n = \mathbb{C}^n$ we mean a map of the following form

$$f : \mathbb{A}^n \to \mathbb{A}^n, a \to f(a) = (f_1(a_1, ..., a_n), ..., f_n(a_1, ..., a_n)),$$

where $f_1, ..., f_n \in \mathbb{C}[x_1, ..., x_n]$ are polynomials and we use the notation $f = (f_1, ..., f_n)$. More generally, suppose X and Y are closed subvarieties of $\mathbb{A}^n$ and $\mathbb{A}^m$ respectively. A regular map f from X to Y has the form $f = (f_1, ..., f_m)$ where the f_i are in the coordinate ring $O(X) = \mathbb{C}[x_1, ..., x_n]/I$ where I is the ideal which defines X, such that the image $f(X)$ lies in Y.

An *automorphism* of X is an endomorphism that admits an inverse which is an endomorphism too. We denote by $\mathrm{Aut}(X)$ the group of automorphisms and by $\mathrm{End}(X)$ the semigroup of endomorphisms of X. A special case is $X \cong \mathbb{A}^n$. One defines the degree of $f \in \mathrm{End}(\mathbb{A}^n)$ as $\deg f := \max_i \deg f_i$. By Aff_n we denote the group of affine transformations of $\mathbb{A}^n$ and by J_n the group of triangular automorphisms (i.e. the automorphisms $(g_1, ..., g_n)$, where $g_i = g_i(x_i, ..., x_n)$ depends only on $x_i, ..., x_n$ for each i). Note that $(g_1, ..., g_n) \in \mathrm{J}_n$ if and only if $g_i = a_i x_i + p_i(x_{i+1}, ..., x_n)$ for all i, where $a_i \in \mathbb{C}^*$ and $p_i \in \mathbb{C}[x_{i+1}, ..., x_n]$. This shows that J_n is, as an ind-variety, isomorphic to

$$(\mathbb{C}^*)^n \times (\mathbb{C} \oplus \mathbb{C}[x_n] \oplus \mathbb{C}[x_{n-1}, x_n] \oplus \cdots \oplus \mathbb{C}[x_2, ..., x_n]).$$

The group $\mathrm{TAut}(\mathbb{A}^n)$ of tame automorphisms is the subgroup of $\mathrm{Aut}(\mathbb{A}^n)$ generated by Aff_n and J_n. If $n = 2$, any automorphism of $\mathbb{A}^n$ is tame. Moreover, $\mathrm{Aut}(\mathbb{A}^2)$ is an amalgamated product of Aff_2 and J_2 with amalgamated subgroup $\mathrm{Aff}_2 \cap \mathrm{J}_2$.

Recently Jean-Philippe Furter and Hanspeter Kraft showed that $\mathrm{Aut}(X)$ has a natural structure of an affine ind-variety for any affine variety X. To show it we start with the following Lemma.

Lemma 1. ([St13, Lemma 3.8]). *Let X and Y be affine varieties. Then the set of morphisms $\mathrm{Mor}(X, Y)$ from X to Y has a canonical structure of an ind-variety.*

Proof. Let $Y \subset \mathbb{A}^n$ and denote by $I \subset O(\mathbb{A}^n)$ the vanishing ideal of Y . The countable dimensional vector space $\mathrm{Mor}(X, \mathbb{A}^n) = O(X)^n$ has the natural structure of an ind-variety by Example 1. It follows, that

$$\mathrm{Mor}(X, Y) = \{f \in \mathrm{Mor}(X, \mathbb{A}^n)|\ \phi \circ f = 0 \text{ for all } \phi \in I\}$$

is closed in $\mathrm{Mor}(X, \mathbb{A}^n)$ and then it has the structure of an ind-variety. One can prove that the ind-structure on $\mathrm{Mor}(X, Y)$ does not depend on the choice of the embedding $Y \subset \mathbb{A}^n$. □

We state without proof the next Lemma.

Lemma 2. *Let X, Y and Z be affine varieties. Then there is a bijection*

$$\begin{aligned} \mathrm{Mor}(X \times Y, Z) &\longleftrightarrow \mathrm{Mor}(X, \mathrm{Mor}(Y, Z)) \\ f &\longmapsto (x \mapsto (y \mapsto f(x, y))) \end{aligned}$$

Moreover, the bijection is an isomorphism of ind-varieties.

Proposition 2. ([St13, Proposition 3.7]). *Let X be an affine variety. Then $\mathrm{Aut}(X)$ has the structure of an ind-group, such that for any algebraic group G, the G-action $G \times X \to X$ corresponds to the ind-group homomorphism $G \to \mathrm{Aut}(X)$.*

Proof. Take any closed embedding $X \subset \mathbb{A}^n$ and let $p : \mathrm{End}(\mathbb{A}^n) \twoheadrightarrow \mathrm{Mor}(X, \mathbb{A}^n)$ be the canonical $\mathbb{C}$-linear projection. Thus $\mathrm{Mor}(X, \mathbb{A}^n) = \varinjlim p(\mathrm{End}(\mathbb{A}^n)_i)$ is filtrated by finite dimensional subspaces and $\mathrm{End}(X) = \varinjlim \mathrm{End}(X)_i$ is an ind-variety, where $\mathrm{End}(X)_i = \mathrm{End}(X) \cap p(\mathrm{End}(\mathbb{A}^n)_i)$. From this construction it follows that $\mathrm{End}(X) \times \mathrm{End}(X) \to \mathrm{End}(X)$, $(f, g) \mapsto f \circ g$ is a morphism and hence $\mathrm{End}(X)$ is an affine ind-semigroup. The set

$$\mathrm{Aut}(X) = \{(f, h) \in \mathrm{End}(X) \times \mathrm{End}(X) | f \circ h = h \circ f = \mathrm{id}\}$$

is closed in $\mathrm{End}(X) \times \mathrm{End}(X)$ and then it has the structure of an ind-variety. As $\mathrm{End}(X)$ is an ind-semigroup, the composition

$$\mathrm{Aut}(X) \times \mathrm{Aut}(X) \to \mathrm{Aut}(X), ((f_1, h_1), (f_2, h_2)) \mapsto (f_1 \circ f_2, h_2 \circ h_1)$$

is a morphism and taking inverses

$$\mathrm{Aut}(X) \to \mathrm{Aut}(X), (f, h) \mapsto (h, f).$$

is a morphism too. Hence, $\mathrm{Aut}(X)$ is an affine ind-group.

Let G be an algebraic group. If $\rho : G \times X \to X$ is a morphism, then $G \to \mathrm{End}(X)$, $g \mapsto \rho_g$ is a morphism by Lemma 2, where $\rho_g : X \to X$ is defined by $\rho_g(x) := \rho(g, x)$. Hence $G \to \mathrm{End}(X) \times \mathrm{End}(X)$, $g \mapsto (\rho_g, \rho_{g^{-1}})$ is a morphism and it induces a homomorphism of ind-groups $G \to \mathrm{Aut}(X)$. Vice versa, if $G \to \mathrm{Aut}(X)$ is a homomorphism of ind-groups, then

$$G \to \mathrm{Aut}(X) \subset \mathrm{End}(X) \times \mathrm{End}(X) \to \mathrm{End}(X)$$

is a morphism and then $G \times X \to X$ is a G-action by Lemma 2. $\square$

Remark 1. In fact, Furter and Kraft showed that the ind-group structure described above is the unique ind-structure on $\mathrm{Aut}(X)$ which satisfies the so-called universal property.

Remark 2. We define the locally closed affine ind-subvariety $\mathrm{Et}(\mathbb{A}^n)$ of $\mathrm{End}(\mathbb{A}^n)$ by the condition that the determinant of the Jacobian matrix $\mathrm{jac}(f)$ of f is in $\mathbb{C}^*$. By the same argument as in the previous proof $\mathrm{Aut}(\mathbb{A}^n)$ is a closed subvariety of $\mathrm{Et}(\mathbb{A}^n)$.

Remark 3. Note that the group of birational transformations $\mathrm{Bir}(\mathbb{P}^n)$ of projective n-space $\mathbb{P}^n$ does not admit a structure of an ind-group (see [BF13]).

Example 2. The automorphism group $\mathrm{Aut}(\mathbb{A}^n)$ of the affine n-space has the structure of an affine ind-group (due to Shafarevich, see [Sha66] and [Sha81]): $\mathrm{Aut}(\mathbb{A}^n) = \varinjlim \mathrm{Aut}(\mathbb{A}^n)_i$, where $\mathrm{Aut}(\mathbb{A}^n)_i$ is the variety of those $g \in \mathrm{Aut}(\mathbb{A}^n)$ which have $\deg g \leq i$. The subgroup $\mathrm{Aff}_n \subset \mathrm{Aut}(\mathbb{A}^n)$ is algebraic and $\mathrm{J}_n \subset \mathrm{Aut}(\mathbb{A}^n)$ is the closed subgroup consisting of algebraic elements, but it is not algebraic itself. Moreover, J_n has the filtration by closed algebraic subgroups. Similarly one turns the endomorphisms $\mathrm{End}(\mathbb{A}^n)$ into an affine ind-monoid through $\mathrm{End}(\mathbb{A}^n) = \varinjlim \mathrm{End}(\mathbb{A}^n)_i$, where $\mathrm{End}(\mathbb{A}^n)_i = \{f \in \mathrm{End}(\mathbb{A}^n) | \deg f \leq i\}$.

3.3. Group actions and vector fields.

3.3.1. *Algebraic group actions.* Let G be an algebraic group which acts on an affine variety X. Then we get a canonical anti-homomorphism of Lie algebras $\xi : \operatorname{Lie} G \to \operatorname{Vec}(X) = \operatorname{Der}(\mathcal{O}(X))$, $A \mapsto \xi_A$, where the vector field ξ_A is defined in the following way (see [Kra11, II.4.4]). Consider the orbit map $\mu_x : G \to X$, $g \mapsto gx$, and set

$$(\xi_A)_x := (d\mu_x)_e(A).$$

We say that $\nu \in \operatorname{Vec}(X)$ is a locally finite vector field if for any $f \in \mathcal{O}(X)$, the vector space generated by $\{\nu^k(f)|k \in \mathbb{N}\}$ is a finite-dimensional vector subspace of $\mathcal{O}(X)$. A vector field $\nu \in \operatorname{Vec}(X)$ is called locally nilpotent if for any $f \in \mathcal{O}(X)$ there exist $k \in \mathbb{N}$ such that $\nu^k(f) = 0$.

3.3.2. *Unipotent elements of* $\operatorname{Aut}(X)$. Let X be an irreducible affine variety. One has a bijective correspondence

$$\operatorname{Aut}(X)_u = \{\text{unipotent elements in } \operatorname{Aut}(X)\} \longleftrightarrow \{G_a - \text{actions on } X\}$$

given in the following way: if $u \in \operatorname{Aut}(X)$ is unipotent, then $G_a = \overline{\langle u \rangle} \subset \operatorname{Aut}(X)$ and we get a G_a-action on X by the homomorphism $G_a \to \operatorname{Aut}(X)$ that sends 1 to u. Conversely, if $\rho : G_a \to \operatorname{Aut}(X)$ is a homomorphism, then $u := \rho(1) \in \operatorname{Aut}(X)$ is unipotent. Additionally, one has a bijective correspondence

$$\{G_a - \text{actions on } X\} \longleftrightarrow \{\text{locally nilpotent vector fields on } X\}$$

which is given in the following way: if $\rho : G_a \times X \to X$ is a G_a-action, then the comorphism $\rho^* : \mathcal{O}(X) \to \mathcal{O}(X)[t]$ induces a derivation $D : \mathcal{O}(X) \to \mathcal{O}(X)$ and the corresponding vector field is locally nilpotent. Vice versa, let ν be a locally nilpotent vector field on X, then it induces a derivation $D : \mathcal{O}(X) \to \mathcal{O}(X)$, $D(f) := d\rho^*(f)|_{t=0}$. Therefore, D induces the map $G_a \to \operatorname{Aut}(X)$, $t \mapsto \operatorname{Exp}(tD)$ which defines a G_a-action on X, where the comorphism of $\operatorname{Exp}(tD)$ is

$$D : \mathcal{O}(X) \to \mathcal{O}(X),\ f \mapsto \sum_{i=0}^{\infty} \frac{t^i}{i!} D^i(f).$$

For more details on the theory of locally nilpotent vector fields see [Fre06].

Let $u \in \operatorname{Aut}(X)$ be unipotent. We denote by $\mathcal{O}(X)^u = \{f(x) \in \mathcal{O}(X)|f(u^{-1}x) = f(x)\}$ the invariant ring of u. If ν is the locally nilpotent vector field that corresponds to u, we have $\mathcal{O}(X)^u = \operatorname{Ker}\nu$. Note that if ν is a locally nilpotent vector field, then $f\nu$ is also locally nilpotent for any $f \in \operatorname{Ker}\nu = \mathcal{O}(X)^u$.

Definition 7. Let $u \in \operatorname{Aut}(X)$ be unipotent and let ν be the corresponding locally nilpotent vector field. For each $f \in \mathcal{O}(X)^u$ we denote by $f \cdot u$ the unipotent automorphism of X corresponding to the locally nilpotent derivation $f\nu$ and we call $f \cdot u$ a modification of u.

The most basic unipotent elements in $\operatorname{Aut}(\mathbb{A}^n)$ are the *translations*, i.e. automorphisms of the form $(x_1 + c_1, ..., x_n + c_n)$ for some $(c_1, ..., c_n) \in \mathbb{C}^n$. A modification of $(x_1, x_2, ..., x_n + 1)$ is an automorphism of the form $(x_1, x_2, ..., x_n + f(x_1, ..., x_{n-1}))$ for some polynomial $f(x_1, ..., x_{n-1})$ which depends only on $x_1, ..., x_{n-1}$.

3.3.3. *Tangent space of* $\operatorname{End}(X)$ *and* $\operatorname{Aut}(X)$. For any $x \in X$ we have a morphism $\mu_x : \operatorname{End}(X) \to X$, $\phi \mapsto \phi(x)$, with differential $d\mu_x : T_e \operatorname{End}(X) \to T_x X$, where $e := \mathrm{id}_X$ is the identity. Thus, for any $H \in T_e \operatorname{End}(X)$, we obtain a vector field ξ_H defined by $(\xi_H)_x = d\mu_x(H)$.

The following result and its proof is due to Furter-Kraft.

Proposition 3. *The tangent space* $T_e \operatorname{End}(X)$ *is canonically isomorphic to the vector fields* $\operatorname{Vec}(X)$*, where the isomorphism is given by* $H \mapsto \xi_H$*.*

Outline of Proof. We choose a closed embedding $X \subset \mathbb{C}^n$ such that $\mathcal{O}(X) = \mathbb{C}[x_1, ..., x_n]/I(X)$. This defines a closed embedding of ind-varieties $\operatorname{End}(X) \subset \operatorname{Mor}(X, \mathbb{C}^n)$, hence $T_{\mathrm{id}}(\operatorname{End}(X)) \subset \mathcal{O}(X)^n$. By definition, $(f_1, ..., f_n) \in \operatorname{End}(X)$ if and only if $F(f_1, ..., f_n) = 0$ for all $F \in I(X)$. Therefore, we have

$$H = (h_1, ..., h_n) \in T_{\mathrm{id}}(\operatorname{End}(X)) \iff F(\overline{x}_1 + \epsilon h_1, ..., \overline{x}_n + \epsilon h_n) = 0 \text{ for all } F \in I(X)$$
$$\iff \sum_{i=1}^{n} h_i \frac{\partial F}{\partial x_i}(\overline{x}_1, ..., \overline{x}_n) = 0 \text{ for all } F \in I(X).$$

The latter means that $\sum_{i=1}^n h_i \frac{\partial}{\partial x_i}$ defines a derivation δ_H of $\mathcal{O}(X)$ by setting $\delta_H = \overline{h}_i$, and every derivation δ of $\mathcal{O}(X)$ arises in this way. Thus, we obtain an isomorphism $T_{\mathrm{id}}(\operatorname{End}(X)) \xrightarrow{\sim} \operatorname{Der}(\mathcal{O}(X)) = \operatorname{Vec}(X)$, given by $H \mapsto \delta_H$. Note that δ_H, as a vector field, is given by $(\delta_H)_x = (h_1(x), ..., h_n(x)) \in T_x X \cong \mathbb{C}^n$.

On the other hand, the morphism $\mu_x : \operatorname{End}(X) \to X$ is given by

$$(f_1, ..., f_n) \mapsto (f_1(x), ..., f_n(x)) \in X \subset \mathbb{C}^n.$$

It follows that

$$\mu_x(\overline{x}_1 + \epsilon h_1(x), ..., \overline{x}_n + \epsilon h_n(x)) = x + \epsilon(h_1(x), ..., h_n(x))$$

for $H = (h_1, ..., h_n) \in T_{\mathrm{id}} \operatorname{End}(X)$. Hence, $(\xi_H)_x = (h_1(x), ..., h_n(x)) = (\delta_H)_x$. □

The following result is due to Hanspeter Kraft.

Proposition 4. *Let* G *be an ind-group which acts on affine variety* X*. Then the map* $\xi : \operatorname{Lie} G \to \operatorname{Vec}(X)$*,* $A \mapsto \xi_A$*, is an anti-homomorphism of Lie algebras. For* $G = \operatorname{Aut}(X)$*, the map* $\xi : \operatorname{Lie} G \to \operatorname{Vec}(X)$ *is injective, so that* $\operatorname{Lie}\operatorname{Aut}(X)$ *can be considered as a Lie subalgebra of* $\operatorname{Vec}(X)$*.*

In the following we will always identify $\operatorname{Lie}\operatorname{Aut}(X)$ with its image in $\operatorname{Vec}(X)$. Note that $\operatorname{Lie}\operatorname{Aut}(X)$ contains all locally finite vector fields. Indeed, if δ is a locally finite vector field of $\operatorname{Vec}(X)$, then there exists an algebraic subgroup G of $\operatorname{Aut}(X)$ such that $\delta \in \operatorname{Lie} G$ (see [CD03]). On the other hand, it is unknown and it is a very interesting problem, whether $\operatorname{Lie}\operatorname{Aut}(X)$ is generated by locally finite vector fields if $\operatorname{Aut}(X)$ is generated by algebraic subgroups.

3.3.4. *The case of* $\operatorname{Aut}(\mathbb{A}^n)$. In this section we are going to compute $\operatorname{Lie}\operatorname{Aut}(\mathbb{A}^n)$. The vector fields on $\mathbb{A}^n$ have the following form: $\operatorname{Vec}(\mathbb{A}^n) = \operatorname{Der}(\mathbb{C}[x_1, ..., x_n]) = \{f_1\partial_1 + \cdots + f_n\partial_n | f_i \in \mathbb{C}[x_1, ..., x_n]\}$, where $\partial_i := \frac{\partial}{\partial x_i}$. Recall that the *divergence* of a vector field $\delta = \sum_{i=1}^n p_i \frac{\partial}{\partial x_i}$ is defined by $\operatorname{Div} \delta := \sum_{i=1}^n \frac{\partial p_i}{\partial x_i}$. We define

$$\operatorname{Vec}^0(\mathbb{A}^n) = \{\delta \in \operatorname{Vec}(\mathbb{A}^n) | \operatorname{Div} \delta = 0\}.$$

and

$$\operatorname{Vec}^c(\mathbb{A}^n) = \{\delta \in \operatorname{Vec}(\mathbb{A}^n) | \operatorname{Div} \delta \in \mathbb{C}\}.$$

Note that both $\operatorname{Vec}^0(\mathbb{A}^n)$ and $\operatorname{Vec}^c(\mathbb{A}^n)$ are Lie subalgebras of $\operatorname{Vec}(\mathbb{A}^n)$ because

$$\operatorname{Div}([\nu, \mu]) = \nu(\operatorname{Div} \mu) - \mu(\operatorname{Div} \nu),$$

where $\nu, \mu \in \operatorname{Vec}(\mathbb{A}^n)$.

The following Lemma can be found in [Sha81].

Lemma 3. *The Lie algebra* $\mathrm{Vec}^0(\mathbb{A}^n)$ *is generated by locally nilpotent vector fields of the form* $m_i\frac{\partial}{\partial x_i}$ *where* m_i *is a monomial in the* x_j *with* $j \neq i$*. Moreover,* $\mathrm{Vec}^0(\mathbb{A}^n)$ *is a simple Lie algebra.*

Proof. (Hanspeter Kraft) (a) If $m = x_1^{k_1}x_1^{k_2}\cdots x_n^{k_n}$ is a monomial we set $m_i := m/x_i^{k_i}$, for $i = 1, ..., n$. Hence,

$$[x_i^{k_i}\frac{\partial}{\partial x_j}, m_i\frac{\partial}{\partial x_i}] = \frac{\partial m}{\partial x_j}\frac{\partial}{\partial x_i} - \frac{\partial m}{\partial x_i}\frac{\partial}{\partial x_j},$$

where $j \neq i$. It follows that for a given $\xi = \sum_{i=1}^n f_i\frac{\partial}{\partial x_i} \in \mathrm{Vec}^0(\mathbb{A}^n)$ we can find a linear combination of the brackets $[x_i^{k_i}\frac{\partial}{\partial x_j}, m_i\frac{\partial}{\partial x_i}]$ which is of the form $\xi' = \sum_{i=1, i\neq j}^n f_i\frac{\partial}{\partial x_i} - h_j\frac{\partial}{\partial x_j}$. Then $\xi - \xi' = h_j\frac{\partial}{\partial x_j}$. Since $\mathrm{Div}(\xi - \xi') = 0$ we see that h_j does not depend on x_j, and so $h_j\frac{\partial}{\partial x_j}$ is a sum of vector fields of the form $c_jm_j\frac{\partial}{\partial x_j}$, where $c_j \in \mathbb{C}^*$.

(b) Let $I \subset \mathrm{Vec}^0(\mathbb{A}^n)$ be a nonzero ideal. If $\xi = \sum_{i=1}^n f_i\frac{\partial}{\partial x_i} \in I$, then $[\frac{\partial}{\partial x_j}, \xi] = \sum_{i=1}^n \frac{\partial f_i}{\partial x_j}\frac{\partial}{\partial x_i} \in I$. It follows that $\frac{\partial}{\partial x_k} \in I$ for some k, and so $\frac{\partial}{\partial x_i} \in I$ for all i, because $[x_k\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_k}] = -\frac{\partial}{\partial x_i}$. If m_i is a monomial which does not depend on x_i, then $[m_i\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_k}] = -\frac{\partial m_i}{\partial x_k}\frac{\partial}{\partial x_i} \in I$. Hence, $I = \mathrm{Vec}^0(\mathbb{A}^n)$ by (a). $\square$

Note that $\mathrm{Vec}^c(\mathbb{A}^n) = \mathrm{Vec}^0(\mathbb{A}^n) \oplus \mathbb{C}E$, where $E := x_1\frac{\partial}{\partial x_1} + \cdots + x_n\frac{\partial}{\partial x_n}$. In fact, E is a locally finite vector field.

The following result and its proof is due to Hanspeter Kraft.

Proposition 5. *The map* ξ *induces an anti-isomorphism of Lie algebras*

$$\mathrm{Lie}\,\mathrm{Aut}(\mathbb{A}^n) \to \mathrm{Vec}^c(\mathbb{A}^n) := \{\delta \in \mathrm{Vec}(\mathbb{A}^n) |\, \mathrm{Div}\,\delta \in \mathbb{C}\}.$$

Proof. We note first that by Remark 2, $\mathrm{Aut}(\mathbb{A}^n)$ is a closed subvariety of $\mathrm{Et}(\mathbb{A}^n)$. It is not difficult to see that $\mathrm{Et}(\mathbb{A}^n)$ is an ind-subvariety of $\mathrm{End}(\mathbb{A}^n)$. This shows that

$$\mathrm{Lie}\,\mathrm{Aut}(\mathbb{A}^n) \subset T_e\,\mathrm{Et}(\mathbb{A}^n) = T_e\{f \in \mathrm{End}(\mathbb{A}^n) |\, \mathrm{jac}(f) \in \mathbb{C}^*\}.$$

For $H = (p_1, ..., p_n) \in \mathrm{End}(\mathbb{A}^n)$ we have $\mathrm{jac}(\mathrm{id} + \epsilon H) = 1 + \epsilon\sum_i \frac{\partial p_i}{\partial x_i} \bmod \epsilon^2$, hence $T_e\,\mathrm{Et}(\mathbb{A}^n) = \mathrm{Vec}^c(\mathbb{A}^n)$. Now it suffices to remark that $\mathrm{Vec}^0(\mathbb{A}^n)$ is generated by locally nilpotent vector fields and that E is locally finite. This proves the claim. $\square$

3.4. **Lie algebra of** $\mathrm{Aut}(X)$ **and action of** $\mathrm{Aut}(\mathbb{A}^n)$ **on** $\mathrm{Vec}(\mathbb{A}^n)$**.**

Theorem 17. *Let* G *be a connected ind-group. If* $\mathrm{Lie}\,G$ *is a simple Lie algebra, then any homomorphism* $F : G \to H$ *of ind-groups is either trivial or the kernel is a discrete subgroup contained in the center of* G*.*

Proof. Let $G = \cup G_i$. By definition, $\mathrm{Lie}\,G = \cup T_eG_i$ and since $\mathrm{Lie}\,G$ is simple, $(dF)_e : \mathrm{Lie}\,G \to \mathrm{Lie}\,H$ is either trivial or injective. If $(dF)_e$ is trivial, the restriction of F to each G_i is a constant map, therefore F is trivial (because G is connected). If $(dF)_e$ is injective, F has discrete kernel K. Then G acts on K by conjugation. Since G is connected it follows that K is included into the center $Z(G)$ of G. $\square$

The Lie algebra of $\mathrm{SAut}(\mathbb{A}^n)$ which is isomorphic to $\mathrm{Vec}^0(\mathbb{A}^n)$ is a simple Lie algebra. But we do not know whether this implies simplicity of the ind-group $\mathrm{SAut}(\mathbb{A}^n)$ i.e. whether there exists a nontrivial closed normal subgroup of $\mathrm{SAut}(\mathbb{A}^n)$. Moreover, KRAFT recently proved that any nontrivial ind-homomorphism from $\mathrm{SAut}(\mathbb{A}^n)$ to an ind-group H is either trivial or is a closed immersion (see [Kra15, Theorem 1.4]). Note that in [Dan74] (see also [FL10]) it was shown that group $\mathrm{SAut}(\mathbb{A}^2)$ is not simple as an abstract group.

3.5. **Characterization of affine varieties.** As we have mentioned in Section 3.2, $\mathrm{End}(X)$ and $\mathrm{Aut}(X)$ have the structure of an ind-semigroup and an ind-group respectively for any affine variety X. Recently HANSPETER KRAFT showed the following result.

Proposition 6. *Let X, Y be affine varieties. Assume that we have an isomorphism $\mathrm{End}(X) \cong \mathrm{End}(Y)$ of ind-semigroups. Then $X \cong Y$.*

Proof. For $x \in X$ denote by $\gamma_x \in \mathrm{End}(X)$ the constant map with value x. Then the map $\iota_X : X \to \mathrm{End}(X)$, $x \mapsto \gamma_x$, is a closed immersion. In fact, it is clearly a morphism, and there is a retraction given by the morphism $\mathrm{ev}_{x_0} : \mathrm{End}(X) \to X$, $\phi \mapsto \phi(x_0)$.

Now we remark that the closed subset $\iota_X(X) \subset \mathrm{End}(X)$ of constant maps is characterized by $\iota_X(X) = \{\phi \in \mathrm{End}(X) | \phi \circ \psi = \phi \text{ for all } \psi \in \mathrm{End}(X)\}$. This implies that every isomorphism of ind-semigroups $\tau : \mathrm{End}(X) \to \mathrm{End}(Y)$ defines a bijective morphism $\tau|_{\iota_X(X)} : \iota_X(X) \to \iota_Y(Y)$. The claim follows since the inverse map is given by $\tau^{-1}|_{\iota_Y(Y)}$. □

A generalization of this result can be found in [AK14], where the authors considered just abstract isomorphism of semigroups of endomorphisms.

On the other hand we can not expect to have such a result if we replace $\mathrm{End}(X)$ by $\mathrm{Aut}(X)$ since for most affine varieties X, $\mathrm{Aut}(X)$ is finite. Recently, HANSPETER KRAFT proved the following result.

Theorem 18. ([Kra15, Theorem 1.1]). *Let Y be a connected affine variety. If $\mathrm{Aut}(\mathbb{A}^n) \cong \mathrm{Aut}(Y)$ as ind-groups, then $Y \cong \mathbb{A}^n$ as varieties.*

Therefore, the affine n-space is determined by its automorphism group in the category of connected affine varieties. There are some futher results in this direction in [Reg15b]. It is of interest to discover more varieties which are determined by their automorphism groups. Moreover, $\mathbb{A}^n$ is also determined by its special automorphism group $U(\mathbb{A}^n)$ in the category of connected affine varieties, where by $U(\mathbb{A}^n)$ we mean the subgroup of $\mathrm{Aut}(X)$ generated by all closed subgroups U such that $U \cong \mathbb{C}^+$. Note that $U(X)$ is not necessarily an ind-group, i.e. $U(X)$ is not necessarily closed in $\mathrm{Aut}(X)$. By an algebraic isomorphism $\phi : U(X) \to U(Y)$ we mean an abstract isomorphism of abstract groups such that the restriction of ϕ to any closed one-dimensional unipotent subgroup is an isomorphism of algebraic groups.

Theorem 19. *Let Y be a connected affine variety. If $U(\mathbb{A}^n)$ and $U(Y)$ are algebraically isomorphic, then $Y \cong \mathbb{A}^n$ as varieties.*

4. Outline of the articles.

4.1. **Automorphisms of the Lie Algebra of Vector Fields on Affine n-Space.** In this section we describe the results of the joint paper [KReg15] with Hanspeter Kraft and give some ideas of the proofs.

The group $\mathrm{Aut}(\mathbb{A}^n)$ acts on $\mathrm{Vec}(\mathbb{A}^n)$ in the usual way. For $\phi \in \mathrm{Aut}(\mathbb{A}^n)$ and $\delta \in \mathrm{Vec}(\mathbb{A}^n)$ we define

$$\mathrm{Ad}(\phi)\delta := \phi^{*-1} \circ \delta \circ \phi^*,$$

where we consider δ as a derivation $\delta\colon \mathbb{C}[x_1, ..., x_n] \to \mathbb{C}[x_1, ..., x_n]$ and $\phi^*\colon \mathbb{C}[x_1, ..., x_n] \to \mathbb{C}[x_1, ..., x_n]$, $f \mapsto f \circ \phi$, is the co-morphism of ϕ. It is shown in [Kul92] that $\mathrm{Ad}\colon \mathrm{Aut}(\mathbb{A}^n) \to \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Vec}(\mathbb{A}^n))$ is an isomorphism, where $\mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Vec}(\mathbb{A}^n))$ denotes the group of automorphisms of the Lie algebra $\mathrm{Vec}(\mathbb{A}^n)$.

In more geometric terms, considering δ as a section of the tangent bundle $T\mathbb{A}^n = \mathbb{A}^n \times \mathbb{C}^n \to \mathbb{A}^n$, one defines the pull-back of δ by

$$\phi^*(\delta) := (d\phi)^{-1} \circ \delta \circ \phi, \text{ i.e., } \phi^*(\delta)_a = (d\phi_a)^{-1}(\delta_{\phi(a)}) \text{ for } a \in \mathbb{A}^n.$$

Clearly, $\phi^*(\delta) = \mathrm{Ad}(\phi^{-1})\delta$. However, the second formula above shows the well-known fact that the pull-back $\phi^*(\delta)$ of a vector field δ is also defined for an étale morphism $\phi : \mathbb{A}^n \to \mathbb{A}^n$. More precisely, let $\phi : \mathbb{A}^n \to \mathbb{A}^n$ be an étale morphism. For any vector field $\delta \in \mathrm{Vec}(\mathbb{A}^n)$ there is a vector field $\phi^*(\delta) \in \mathrm{Vec}(\mathbb{A}^n)$ defined by $\phi^*(\delta)_a := (d\phi)_a^{-1}\delta_{\phi(a)}$ for $a \in \mathbb{A}^n$. It is uniquely determined by $\phi^*(\delta)\phi^*(f) = \phi^*(\delta f)$ for $f \in \mathbb{C}[x_1, ..., x_n]$. The map $\phi^* : \mathrm{Vec}(\mathbb{A}^n) \to \mathrm{Vec}(\mathbb{A}^n)$ is an injective homomorphism of Lie algebras satisfying $\phi^*(h\delta) = \phi^*(h)\phi^*(\delta)$ for $h \in \mathbb{C}[x_1, ..., x_n]$. Moreover, $(\eta \circ \phi)^* = \phi^* \circ \eta^*$.

First, we give a short proof of the fact that $\mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Vec}(\mathbb{C}[x_1, ..., x_n])) = \mathrm{Aut}(\mathbb{A}^n)$ in [KReg15, Theorem 3.1]. In order to prove this we first note that the map

$$\mathrm{Ad} : \mathrm{Aut}(\mathbb{A}^n) \to \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Vec}(\mathbb{A}^n))$$

is injective. To show surjectivity we consider the subgroup $S = (\mathbb{C}^+)^n \subset \mathrm{Aff}_n$ of translations. Then $\mathfrak{s} := \mathrm{Lie}\, S = \langle \partial_{x_1}, ..., \partial_{x_n} \rangle$. Let $\theta : \mathrm{Vec}(\mathbb{A}^n) \xrightarrow{\sim} \mathrm{Vec}(\mathbb{A}^n)$ be an isomorphism and $\theta(\mathfrak{s}) = \mathfrak{u}$. We show that $\mathfrak{u}$ is generated by locally nilpotent vector fields too. Therefore, $\mathfrak{u}$ is a Lie algebra of some unipotent subgroup $U \subset \mathrm{Aut}(\mathbb{A}^n)$.

Because $\mathfrak{cent}_{\mathrm{Vec}(\mathbb{A}^n)}(\mathfrak{s}) = \mathfrak{s}$ it follows that $\mathfrak{cent}_{\mathrm{Vec}(\mathbb{A}^n)}(\mathfrak{u}) = \mathfrak{u}$. By using this, we show that the orbit maps $\mu_S : S \xrightarrow{\sim} \mathbb{A}^n$ and $\mu_U : U \xrightarrow{\sim} \mathbb{A}^n$ are isomorphisms. Then one sees that $\phi := \mu_S \circ \psi \circ \mu_U^{-1}$ has the property that $\phi \circ u \circ \phi^{-1} = \psi(u)$ for all $u \in U$. Hence, the automorphism $\theta' := \mathrm{Ad}(\phi) \circ \theta \in \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Vec}(\mathbb{A}^n))$ sends $\mathrm{Lie}(S)$ isomorphically onto itself. Then, one proves that there is an $\alpha \in \mathrm{Aff}_n$ such that $\mathrm{Ad}(\alpha) \circ \theta'$ is the identity on $\mathrm{Lie}(\mathrm{Aff}_n)$. From here we finish the proof by showing that in case θ is the identity on $\mathrm{Lie}(\mathrm{Aff}_n)$, it is the identity on $\mathrm{Vec}(\mathbb{A}^n)$.

The aim of [KReg15] is to prove the following result about the automorphism groups of Lie algebras $\mathrm{Vec}^0(\mathbb{A}^n)$ and $\mathrm{Vec}^c(\mathbb{A}^n)$.

Theorem 20. [KReg15, Main Theorem] *There are canonical isomorphisms of groups*

$$\mathrm{Aut}(\mathbb{A}^n) \cong \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Vec}(\mathbb{A}^n)) \cong \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Vec}^c(\mathbb{A}^n)) \cong \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Vec}^0(\mathbb{A}^n)).$$

Remark 4. (a) The theorem above holds over any field K of characteristic zero.

(b) Hanspeter Kraft showed that the groups in Theorem 20 have a natural structure of ind-groups and that the maps are all isomorphisms of ind-groups.

To prove Theorem 20 it is enough to show that $\mathrm{Aut}(\mathrm{Vec}^0(\mathbb{A}^n)) \cong \mathrm{Aut}(\mathbb{A}^n)$. In order to do so, we first show that the canonical map

$$\mathrm{Ad} : \mathrm{Aut}(\mathbb{A}^n) \to \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Vec}^0(\mathbb{A}^n)) \tag{1}$$

is injective. Therefore, it is enough to show surjectivity of Ad.

Recall that a Darboux polynomial of δ is a nonconstant polynomial $f \in \mathbb{C}[x_1, ..., x_n]$ such that $\delta(f) = hf$ for some $h \in \mathbb{C}[x_1, ..., x_n]$.

If $n = 1$ it is easy to see that Ad from (1) is surjective, hence we can assume that $n \geq 2$. Let θ be an automorphism of the Lie algebra $\mathrm{Vec}^0(\mathbb{A}^n)$. Put $\delta_i := \theta(\partial_{x_i})$. Then the vector fields $\delta_1, ..., \delta_n$ are pairwise commuting and $\mathbb{C}$-linearly independent. Since ∂_{x_i} acts locally nilpotently on $\mathrm{Vec}^0(\mathbb{A}^n)$, the same holds for δ_i.

In the following we will use vector fields with rational coefficients:

$$\mathrm{Vec}^{rat}(\mathbb{A}^n) := \mathbb{C}(x_1, ..., x_n) \otimes_{\mathbb{C}[x_1,...,x_n]} \mathrm{Vec}(\mathbb{A}^n) = \bigoplus_{i=1}^{n} \mathbb{C}(x_1, ..., x_n)\partial_{x_i}.$$

We first show that the $\delta_1, ..., \delta_n$ do not have a common Darboux polynomial. Hence one shows that there is an étale morphism $\phi : \mathbb{A}^n \to \mathbb{A}^n$ with $\delta_i = \phi^*(\partial_{x_i})$ for all i. Then the composition $\theta' := \theta^{-1} \circ \phi^* : \mathrm{Vec}^0(\mathbb{A}^n) \to \mathrm{Vec}^0(\mathbb{A}^n)$ is an injective homomorphism of Lie algebras and $\theta'(\partial_{x_i}) = \partial_{x_i}$. Hence, Lemma 5.4 from [KReg15] implies that $\theta' = \mathrm{Ad}(s) = (s^{-1})^*$, where $s \in \mathrm{Aut}(\mathbb{A}^n)$ is a translation, hence $\theta = (\phi \circ s)^*$. Now we show that $\psi := \phi \circ s$ is an automorphism of $\mathbb{A}^n$, and so $\theta = \mathrm{Ad}(\psi^{-1})$ as claimed.

As a consequence of Theorem 20 we get the following result which is due to Kulikov, (see [Kul92, Theorem 4] cf. [KReg15, Corollary 4.4]).

Corollary 1. *If every injective endomorphism of the Lie algebra* $\mathrm{Vec}(\mathbb{A}^n)$ *is an automorphism, then the Jacobian Conjecture holds in dimension* n*.*

Remark 5. In fact, one can show that if every injective endomorphism of the Lie algebra $\mathrm{Vec}^0(\mathbb{A}^n)$ is an automorphism, then the Jacobian Conjecture holds in dimension n.

Remark 6. It was proved by Belov-Kanel and Yu that every automorphism of $\mathrm{Aut}(\mathbb{A}^n)$ as an ind-group is inner (see [BYu12]). Using Theorem 20 and Remark 4(b), one can give a short proof of this and extend it to the closed subgroup $\mathrm{SAut}(\mathbb{A}^n) \subset \mathrm{Aut}(\mathbb{A}^n)$ of automorphisms with Jacobian determinant equal to 1.

4.2. **Lie subalgebras of plane vector fields and the jacobian conjecture.** In this section we describe the main results of the paper [Reg15a] and indicate some ideas of the proofs.

It is a well-known consequence of the amalgamated product structure of $\mathrm{Aut}(\mathbb{A}^2)$ that every reductive subgroup $G \subset \mathrm{Aut}(\mathbb{A}^2)$ is conjugate to a subgroup of $\mathrm{GL}_2(\mathbb{C}) \subset \mathrm{Aut}(\mathbb{A}^2)$, i.e. there is a $\psi \in \mathrm{Aut}(\mathbb{A}^2)$ such that $\psi G \psi^{-1} \subset \mathrm{GL}_2(\mathbb{C})$ ([Kam79], cf. [Kr96]). The "Linearization Problem" asks whether the same holds for $\mathrm{Aut}(\mathbb{A}^n)$. It was shown by Schwarz in [Sch89] that this is not the case in dimensions $n \geq 4$ (cf. [Kn91]).

In [Reg15a] we consider the analogue of the Linearization Problem for Lie algebras. By Proposition 5 the Lie algebra $\mathrm{Lie}(\mathrm{Aut}(\mathbb{A}^2))$ is canonically isomorphic to the Lie algebra $\mathrm{Vec}^c(\mathbb{A}^2)$. The Lie subalgebra

$$L := \mathbb{C}(x^2\partial_x - 2xy\partial_y) \oplus \mathbb{C}(x\partial_x - y\partial_y) \oplus \mathbb{C}\partial_x \subset \mathrm{Vec}^0(\mathbb{A}^2) \subset \mathrm{Vec}^c(\mathbb{A}^2),$$

where $\partial_x := \frac{\partial}{\partial x}$ and $\partial_y := \frac{\partial}{\partial y}$, is isomorphic to $\mathfrak{sl}_2$, but not conjugate to the standard $\mathfrak{sl}_2 := \langle x\partial_y, y\partial_x, x\partial_x - y\partial_y\rangle \subset \mathrm{Vec}^c(\mathbb{A}^2)$ under $\mathrm{Aut}(\mathbb{A}^2)$ (see [Reg15a, Remark 4.2]). This shows that the Linearization Problem for $\mathrm{Lie}\,\mathrm{Aut}(\mathbb{A}^2)$ does not hold. However, for some other Lie subalgebras of $\mathrm{Vec}^c(\mathbb{A}^2)$ the situation is different. Let $\mathrm{Aff}_2(\mathbb{C}) \subset \mathrm{Aut}(\mathbb{A}^2)$ be the group of affine transformations and $\mathrm{SAff}_2(\mathbb{C}) \subset \mathrm{Aff}_2(\mathbb{C})$ the subgroup of affine transformations with determinant equal to 1, and denote by $\mathrm{aff}_2 = \langle \partial_x, \partial_y, x\partial_x, y\partial_y, x\partial_y, y\partial_x\rangle$, respectively $\mathrm{saff}_2 = \langle \partial_x, \partial_y, x\partial_y, y\partial_x, x\partial_x - y\partial_y\rangle$ their Lie algebras. The first result we prove is the following (see [Reg15a, Proposition 3.6]). For $f \in \mathbb{C}[x,y]$ we set $D_f := f_x\partial_y - f_y\partial_x \in \mathrm{Vec}^c(\mathbb{A}^2)$. Note that every vector field with divergence 0 has this form.

Theorem 21. *Let $L \subset \mathrm{Vec}^c(\mathbb{A}^2)$ be a Lie subalgebra isomorphic to saff_2. Then there is an étale map $\phi\colon \mathbb{A}^2 \to \mathbb{A}^2$ such that $L = \phi^*(\mathrm{saff}_2)$. More precisely, if (D_f, D_g) is a basis of the solvable radical of L, then*

$$L = \langle D_f, D_g, D_{f^2}, D_{g^2}, D_{fg}\rangle,$$

and one can take $\phi = (f, g)$.

In order to prove this result we introduce the Poisson algebra P as the Lie algebra with underlying vector space $\mathbb{C}[x,y]$ and with Lie bracket $\{f,g\} := j(f,g) = f_x g_y - f_y g_x$ for $f, g \in P$.

There is a canonical surjective homomorphism of Lie algebras $\mu : P \to \mathrm{Vec}^0(\mathbb{A}^2)$, $h \to D_h := h_x\partial_y - h_y\partial_x$, with kernel $\ker\mu = \mathbb{C}$. For $f, g \in \mathbb{C}[x,y]$ such that $\{f,g\} \in \mathbb{C}^*$ we put

$$P_{f,g} := \langle 1, f, g, f^2, fg, g^2\rangle \subset P.$$

This Lie algebra is isomorphic to $P_{\leq 2} := \langle 1, x, y, x^2, xy, y^2\rangle$. Clearly, $P_{f,g} = P_{f_1,g_1}$ if $\langle 1, f, g\rangle = \langle 1, f_1, g_1\rangle$. Denoting by $\mathrm{rad}\,L$ the solvable radical of the Lie algebra L we get $\mathrm{rad}\,P_{f,g} = \langle 1, f, g\rangle$ and $P_{f,g}/\mathrm{rad}\,P_{f,g} \cong \mathfrak{sl}_2$. Then we show (see [KReg15, Proposition 2.8]) that each subalgebra of P isomorphic to $P_{\leq 2}$ is equal to $P_{f,g}$ for some $f, g \in \mathbb{C}[x,y]$, where $\{f,g\} \in \mathbb{C}^*$. The proof is based on the fact that we can easily compute the centralizer $\mathfrak{cent}_P(f)$ of $f \in P$ and then by using defining relations of $P_{\leq 2}$ conclude the result.

Now let $L \subset \mathrm{Vec}^c(\mathbb{A}^2)$ be a Lie subalgebra isomorphic to saff_2. Then $L = [L, L] \subset [\mathrm{Vec}^c(\mathbb{A}^n), \mathrm{Vec}^c(\mathbb{A}^n)] = \mathrm{Vec}^0(\mathbb{A}^n)$. Hence, we show that it suffices to check that $Q := \mu^{-1}(L) \subset P$ is isomorphic to $P_{\leq 2}$. From this one shows that each subalgebra of $\mathrm{Vec}^0(\mathbb{A}^2)$ isomorphic to $\mathrm{saff}_2 = \mu(P_{\leq 2})$ is equal to $L_{f,g} = \langle D_f, D_g, D_{f^2}, D_{g^2}, D_{fg}\rangle = \phi^*(\mathrm{saff}_2)$, where $\phi = (f,g) : \mathbb{A}^2 \to \mathbb{A}^2$ is an étale map.

We can extend Theorem 21 to the following result.

Theorem 22. *Let $L \subset \mathrm{Vec}^c(\mathbb{A}^2)$ be a Lie subalgebra isomorphic to aff_2. Then there is an étale map $\phi\colon \mathbb{A}^2 \to \mathbb{A}^2$ such that $L = \phi^*(\mathrm{aff}_2)$. More precisely, if (D_f, D_g) is a basis of the solvable radical of $[L, L]$, then*

$$L = \langle D_f, D_g, fD_f, gD_g, fD_g, gD_f\rangle,$$

and one can take $\phi = (f, g)$.

Let $L \subset \mathrm{Vec}^c(\mathbb{A}^2)$ be isomorphic to aff_2. Then $L = [L, L] \oplus \mathbb{C}D$ for some $D \in \mathrm{Vec}^c(\mathbb{A}^n)$ and $[L, L] \cong \mathrm{saff}_2 \subset [\mathrm{Vec}^c(\mathbb{A}^n), \mathrm{Vec}^c(\mathbb{A}^n)] = \mathrm{Vec}^0(\mathbb{A}^n)$. Therefore, $L = \phi^*(\mathrm{saff}_2) \oplus \mathbb{C}D$ for some étale map $\phi : \mathbb{A}^2 \to \mathbb{A}^2$. We claim that $\phi^*(\mathrm{aff}_2) = L$. To show this we first note that $\phi^*(\mathrm{aff}_2) = L_{f,g} \oplus \mathbb{C}E$, where E is the image of the Euler element of aff_2. Since $\mathrm{Vec}^c(\mathbb{A}^2) = \mathrm{Vec}^0(\mathbb{A}^2) \oplus \mathbb{C}D'$ for any $D' \in \mathrm{Vec}^c(\mathbb{A}^2)$

with $\operatorname{Div} D' \neq 0$, we can write $D = aE + F$ with some $a \in \mathbb{C}^*$ and $F \in \operatorname{Vec}^0(\mathbb{A}^2)$ i.e., $F = D_h$ for some $h \in \mathbb{C}[x, y]$. By construction, $F = D - aE$ commutes with $M := \langle D_{f^2}, D_{g^2}, D_{fg}\rangle \cong \mathfrak{sl}_2$. Hence, we get $\{h, f^2\} = c$, where $c \in \mathbb{C}$. Thus $c = \{h, f^2\} = 2f\{h, f\}$ which implies that $\{h, f\} = 0$. Similarly, we find $\{h, g\} = 0$, hence it is not difficult to see that $h \in \mathbb{C}$ and so $D_h = 0$ which implies $D = aE$ and the proof follows.

As a consequence of the classification above, we obtain the next result (see [Reg15a, Theorem 4.1, Corollary 4.4]). Recall that a Lie subalgebra of $\operatorname{Vec}(\mathbb{A}^2)$ is *algebraic* if it acts locally finitely on $\operatorname{Vec}(\mathbb{A}^2)$.

Theorem 23. *The following statements are equivalent:*
(i) *The Jacobian Conjecture holds in dimension 2;*
(ii) *All Lie subalgebras $L \subset \operatorname{Vec}^c(\mathbb{A}^2)$ isomorphic to $\mathfrak{saff}_2$ are conjugate under $\operatorname{Aut}(\mathbb{A}^2)$;*
(iii) *All Lie subalgebras $L \subset \operatorname{Vec}^c(\mathbb{A}^2)$ isomorphic to $\mathfrak{aff}_2$ are conjugate under $\operatorname{Aut}(\mathbb{A}^2)$;*
(iv) *All Lie subalgebras $L \subset \operatorname{Vec}^c(\mathbb{A}^2)$ isomorphic to $\mathfrak{aff}_2$ are algebraic;*
(v) *All Lie subalgebras $L \subset \operatorname{Vec}^c(\mathbb{A}^2)$ isomorphic to $\mathfrak{saff}_2$ are algebraic.*

The implication $(i) \Rightarrow (ii)$ is easy and follows from Theorem 21. To show the implication $(ii) \Rightarrow (iii)$ we consider a Lie subalgebra $L \subset \operatorname{Vec}^c(\mathbb{A}^2)$ isomorphic to $\mathfrak{aff}_2$, and set $L' := [L, L] \cong \mathfrak{saff}_2$. By (ii) there is an automorphism $\phi \in \operatorname{Aut}(\mathbb{A}^2)$ such that $L' = \phi^*(\mathfrak{saff}_2)$. It follows that $\phi^*(\mathfrak{aff}_2) = L$ since L is determined by $\operatorname{rad}(L')$ as a Lie subalgebra, by [Reg15a, Proposition 3.9].

To show the implication $(iii) \Rightarrow (iv)$ we consider a Lie subalgebra $L \subset \operatorname{Vec}^c(\mathbb{A}^2)$ isomorphic to $\mathfrak{aff}_2$. Then by (iii) $L = \phi^*(\mathfrak{aff}_2)$ for some $\phi \in \operatorname{Aut}(\mathbb{A}^n)$. Hence, $L = \operatorname{Lie} \phi(\operatorname{Aff}_2(\mathbb{C}))$ and the claim follows. The implication $(iv) \Rightarrow (v)$ one can show by using the fact that $\mathfrak{saff}_2 = [\mathfrak{aff}_2, \mathfrak{aff}_2]$.

Assume (v) holds. Then any $L \subset \operatorname{Vec}^c(\mathbb{A}^2)$ isomorphic to $\mathfrak{saff}_2$ is equal to $\operatorname{Lie} G$, where $\operatorname{SAff}(\mathbb{C}) \cong G \subset \operatorname{Aut}(\mathbb{A}^2)$. Since there is a subgroup H of G isomorphic to $\operatorname{SL}_2(\mathbb{C})$, we show that (i) follows from the fact that all subgroups of $\operatorname{Aut}(\mathbb{A}^2)$ isomorphic to $\operatorname{SL}_2(\mathbb{C})$ are conjugate.

4.3. **Characterization of n-dimensional SL_n-varieties.** In this section we give the main results of the paper [Reg15b] and some ideas of the proofs.

In the joint paper [KRZ15] we show that for n 3 a normal affine SL_n-variety of dimension n is isomorphic to a quotient $\mathbb{A}^n_d := \mathbb{C}^n/\mu_d$ where the cyclic group $\mu_d := \{\xi \in \mathbb{C}^* | \xi^d = 1\}$ acts by scalar multiplication on $\mathbb{C}^n$. For $n = 2$ there are two more cases, namely SL_2/T and $\operatorname{SL} 2/N$ where $T \subset \operatorname{SL}_2$ is the torus of the diagonal matrices and $N = N(T)$ is the normalizer of T. The main result of the paper [Reg15b] shows that a normal n- dimensional affine SL_n-variety is determined by its automorphism group. More precisely, we have the following result. Theorem 11. Let X be a normal affine SL_n-variety of dimension n, i.e. $X \cong \mathbb{A}^n_d$, SL_2/T or SL_2/N, and let Y be any normal affine variety. If $Aut(Y)$ is isomorphic to $Aut(X)$ as ind-groups, then Y is isomorphic to X as varieties. Theorem 11 is a special case of the next theorem where we include the case of a non-normal irreducible Y . The coordinate ring of And is given by

$$O(\mathbb{A}^n_d) = \mathbb{C} \oplus \mathbb{C}[x_1, \cdots, x_n]_{kd}$$

If $d \geq 2$, then $0 \in \mathbb{A}^n_d$ is an isolated singularity and so every automorphism of $\mathbb{A}^n_d$ fixes 0. This implies that the non-normal varieties $\mathbb{A}^n_{d,s}$, $s \geq 2$, with coordinate ring

$$O(\mathbb{A}^n_{d,s}) = \mathbb{C} \oplus \mathbb{C}[x_1, ..., x_n]_{kd}$$

and normalization $\eta : \mathbb{A}^n_d \to \mathbb{A}^n_{d,s}$, have the same automorphism group as $\mathbb{A}^n_d$. Here is the main result.

Theorem 24. *Let Y be an irreducible affine variety. (a) If $\mathrm{Aut}(Y) \cong \mathrm{Aut}(\mathbb{A}^n_d)$ as ind-groups, for some $n1, d2$, then $Y \cong A\ltimes_{d,s}$ for some $s \geq 1$.*

(b) If $\mathrm{Aut}(Y) \cong \mathrm{Aut}(\mathrm{SL}\,2/T)$ as ind-groups, then $Y \cong \mathrm{SL}_2/T$ as varieties, and the same holds for $\mathrm{SL}\,2/N$.

We also have some extensions of these results for the special au- tomorphism group $U(X)$ which we will formulate below.

In this paper we show a similar result as in Theorem 18 for a normal irreducible affine n-dimensional SL_n-variety X. It is shown in [KRZ15] that in case $n \geq 3$ any such X is isomorphic to A_d i.e. to the quotient of $\mathbb{C}^n$ by a cyclic group $\mu_d = \{\xi \in \mathbb{C}^* | \xi^d = 1\}$, $d \geq 1$, where the action is given by $\xi \cdot (x_1, ..., x_n) = (\xi x_1, ..., \xi x_n)$. Later on we consider only the case $d > 1$. In case $n = 2$, X can only be isomorphic to SL_2/T, $\mathrm{SL}_2/N(T)$ or 2-dimensional $\mathbb{C}^2/\mu_d$ (see [Reg15b, Lemma 5] cf. [Pop73]), where T is the standard subtorus of SL_2 and $N(T)$ denotes the normalizer of T.

The main result of this paper shows that any normal irreducible affine n-dimensional SL_n-variety is determined by its automorphism group in the category of normal affine irreducible varieties.

Theorem 25. *Let $X = \mathrm{SL}_2/T, \mathrm{SL}_2/N(T)$ or A_d and Y be an irreducible normal affine variety. If $\mathrm{Aut}(Y) \cong \mathrm{Aut}(X)$ as ind-groups, then $Y \cong X$ as varieties.*

In fact, Theorem 25 is a particular case of Theorem 26.

In case Y is not necessarily normal, the situation changes since $\mathrm{Aut}(A_d)$ is canonically isomorphic to $\mathrm{Aut}(A^s_d)$ for any $s \in \mathbb{N}$, where A^s_d is a variety with a ring of regular functions $\mathcal{O}(A^s_d) = \mathbb{C} \oplus \bigoplus_{k=s}^{\infty} \mathbb{C}[x_1, ..., x_n]_{dk}$, where $\mathbb{C}[x_1, ..., x_n]_{dk}$ denotes the homogeneous polynomials of degree dk.

Theorem 26. *Let Y be an irreducible affine variety.*

(a) if $\mathrm{Aut}(Y) \cong \mathrm{Aut}(A_d)$ as ind-groups, then $Y \cong A^s_d$ for some $s \in \mathbb{N}$,

(b) if $X \cong \mathrm{SL}_2/T$ or $X \cong \mathrm{SL}_2/N(T)$ and $\mathrm{Aut}(Y) \cong \mathrm{Aut}(X)$, then $Y \cong X$.

Theorem 26 follows from Theorem 28 if X is different from SL_2/T, $\mathrm{SL}_2/N(T)$, $\mathbb{C}^2/\mu_2$ and $\mathbb{C}^2/\mu_4$. By comparing weights of root subgroups of the automorphisms groups of mentioned varieties with respect to standard subtori, we see that SL_2/T can only be isomorphic to $\mathbb{C}^2/\mu_2$ and $\mathrm{SL}_2/N(T)$ can only be isomorphic to $\mathbb{C}^2/\mu_4$. To distinguish SL_2/T from $\mathbb{C}^2/\mu_2$ by their automorphism groups we remark that $\mathbb{C}^2/\mu_d$ admits a faithfull action of 2-dimensional torus and SL_2/T does not. Analogously, we distinguish $\mathrm{SL}_2/N(T)$ from $\mathbb{C}^2/\mu_4$ by their automorphism groups.

Note that an isomorphism $\phi : \mathrm{Aut}(X) \to \mathrm{Aut}(Y)$ of ind-groups induces an algebraic isomorphism $\phi^u : U(X) \to U(Y)$. In case $U(X)$ and $U(Y)$ are closed subgroups of $\mathrm{Aut}(X)$ and $\mathrm{Aut}(Y)$ respecively, ϕ^u is an isomorphism of ind-groups.

Theorem 25 is extends to the following result.

Theorem 27. *Let Y be an irreducible affine normal variety.*

(a) $U(A^2_2) \cong U(\mathrm{SL}_2/T)$. Moreover, if $U(A^2_2) \cong U(Y)$, then Y is isomorphic either to A^2_2 or to SL_2/T,

(b) $U(A^2_4) \cong U(\mathrm{SL}_2/N(T))$. Moreover, if $U(A^2_4) \cong U(Y)$, then Y is isomorphic either to A^2_4 or to $\mathrm{SL}_2/N(T)$,

(c) Let X be isomorphic to A^2_d, SL_2/T or to $\mathrm{SL}_2/N(T)$ except A^2_4 and A^2_2 and $U(X) \cong U(Y)$, then $Y \cong X$.

If we skip the condition of normality on Y, we get the following result.

Theorem 28. *Let $X = A_d$, SL_2/T or $\mathrm{SL}_2/N(T)$ and Y be an irreducible affine variety. Let also $U(Y)$ and $U(X)$ are algebraically isomorphic. Then*

(a) if $n = 2$ and $X = A_2$ or SL_2/T, then $Y \cong A_2^s$ for some $s \in \mathbb{N}$ or $Y \cong \mathrm{SL}_2/T$,

(b) if $n = 2$ and $X = A_4$ or $\mathrm{SL}_2/N(T)$, then $Y \cong A_4^s$ for some $s \in \mathbb{N}$ or $Y \cong \mathrm{SL}_2/N(T)$,

(c) otherwise, normalization of Y is isomorphic to X and moreover, $Y \cong A_d^s$ for some $s \in \mathbb{N}$.

To prove this theorem, first, we show that all tori of maximal dimension $U(X)$ are congugate, where X is as in Theorem 28. Then by comparing weights of root subgroups of $U(X)$ and $U(Y)$ with respect to standard subtori we conclude the result.

4.4. **Groups of automorphisms of Danielewski surfaces.** In [LR15] we consider Danielewski surfaces $D_p = \{(x, y, z) \in \mathbb{C}^3 | xy = p(z)\}$, where $p(z) \in \mathbb{C}[z]$ is a polynomial of degree ≥ 2 with no multiple roots. The letter implies that D_p is smooth. As an example, we have $\mathrm{SL}_2/T \cong V(xy - z^2 + z) = D_{z(z-1)}$.

Let X be an affine variety. Let us denote by μ_2 the cyclic group of order 2, which acts on $\mathbb{C}^2$ in the following way: $\xi \cdot (x, y) = (\xi x, \xi y)$, where $\xi \in \mu_2$. In [Reg15b, Proposition 10] it is shown that there is an abstract isomorphism $\phi : U(\mathrm{SL}_2/T) \to U(\mathbb{C}^2/\mu_2)$ such that the restriction of ϕ to any algebraic subgroup $U \cong \mathbb{C}^+$ is an isomorphism of algebraic groups. Note that $U(\mathbb{C}^2/\mu_2)$ is a closed subgroup of $\mathrm{Aut}(\mathbb{C}^2/\mu_2)$ (see [Reg15b, Proposition 10]) and $U(\mathrm{SL}_2/T) = \mathrm{Aut}^0(\mathrm{SL}_2/T)$ is a closed subgroup of $\mathrm{Aut}(\mathrm{SL}_2/T)$. Hence, $U(\mathrm{SL}_2/T)$ and $U(\mathbb{C}^2/\mu_2)$ are ind-groups.

Theorem 29. *The ind-groups $U(\mathrm{SL}_2/T)$ and $U(\mathbb{C}^2/\mu_2)$ are not isomorphic.*

To prove this we introduce the Lie subalgebra $\mathrm{Lie}^{\mathrm{alg}}\, U(\mathbb{C}^2/\mu_2)$ of $\mathrm{Vec}(\mathbb{C}^2/\mu_2)$ generated by locally nilpotent vector fields on $\mathbb{C}^2/\mu_2$. By using the fact that $\mathbb{C}^2/\mu_2$ has an isolated singular point, we show that $\mathrm{Lie}^{\mathrm{alg}}\, U(\mathbb{C}^2/\mu_2)$ is not a simple Lie algebra. On the other hand, we show that Lie subalgebra $\mathrm{Lie}^{\mathrm{alg}}\, U(D_p)$ of $\mathrm{Vec}(D_p)$ generated by locally nilpotent vector fields on D_p is simple.

Theorem 30. *Let D_p be a Danielewski surface, where $\deg p \geq 2$. Then $\mathrm{Lie}^{alg}\, U(D_p)$ is a simple Lie algebra.*

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Automorphisms of the Lie Algebra of Vector Fields on Affine n-Space

Hanspeter Kraft Andriy Regeta

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Abstract

We study the vector fields $\mathrm{Vec}(\mathbb{A}^n)$ of affine n-space $\mathbb{A}^n$, the subspace $\mathrm{Vec}^c(\mathbb{A}^n)$ of vector fields with constant divergence, and the subspace $\mathrm{Vec}^0(\mathbb{A}^n)$ of vector fields with divergence zero, and we show that their automorphisms, as Lie algebras, are induced by the automorphisms of $\mathbb{A}^n$:

$$\mathrm{Aut}(\mathbb{A}^n) \xrightarrow{\sim} \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Vec}(\mathbb{A}^n)) \xrightarrow{\sim} \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Vec}^c(\mathbb{A}^n)) \xrightarrow{\sim} \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Vec}^0(\mathbb{A}^n)).$$

This generalizes results of the second author obtained in dimension 2, see [Reg13]. The case of $\mathrm{Vec}(\mathbb{A}^n)$ goes back to KULIKOV [Kul92].

This generalization is crucial in the context of infinite-dimensional algebraic groups, because $\mathrm{Vec}^c(\mathbb{A}^n)$ is canonically isomorphic to the Lie algebra of $\mathrm{Aut}(\mathbb{A}^n)$, and $\mathrm{Vec}^0(\mathbb{A}^n)$ is isomorphic to the Lie algebra of the closed subgroup $\mathrm{SAut}(\mathbb{A}^n) \subset \mathrm{Aut}(\mathbb{A}^n)$ of automorphisms with Jacobian determinant equal to 1.

Keywords. Automorphisms, vector fields, Lie algebras, affine n-space.

1 Introduction

Let K be an algebraically closed field of characteristic zero. Denote by $\mathrm{Vec}(\mathbb{A}^n)$ the Lie algebra of polynomial vector fields on affine n-space $\mathbb{A}^n = K^n$:

$$\mathrm{Vec}(\mathbb{A}^n) = \mathrm{Der}(K[x_1, \ldots, x_n]) = \left\{ \sum_i f_i \frac{\partial}{\partial x_i} \mid f_i \in K[x_1, \ldots, x_n] \right\}.$$

Hanspeter Kraft: Universität Basel, Departement Mathematik und Informatik, Spiegelgasse 1, CH-4051 Basel; e-mail: hanspeter.kraft@unibas.ch

Andriy Regeta: Universität Basel, Departement Mathematik und Informatik, Spiegelgasse 1, CH-4051 Basel; e-mail: andriy.regeta@unibas.ch

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where we use the standard identification of a derivation δ with $\sum_i \delta(x_i)\frac{\partial}{\partial x_i}$. The group $\mathrm{Aut}(\mathbb{A}^n)$ of polynomial automorphisms of $\mathbb{A}^n$ acts on $\mathrm{Vec}(\mathbb{A}^n)$ in the usual way. For $\varphi \in \mathrm{Aut}(\mathbb{A}^n)$ and $\delta \in \mathrm{Vec}(\mathbb{A}^n) = \mathrm{Der}(K[x_1, \ldots, x_n])$ we define

$$\mathrm{Ad}(\varphi)\delta := \varphi^{*-1} \circ \delta \circ \varphi^*$$

where $\varphi^*\colon K[x_1, \ldots, x_n] \to K[x_1, \ldots, x_n]$, $f \mapsto f \circ \varphi$, is the comorphism of φ. It is shown in [Kul92] that $\mathrm{Ad}\colon \mathrm{Aut}(\mathbb{A}^n) \to \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Vec}(\mathbb{A}^n))$ is an isomorphism. We will give a short proof in section 3.

Recall that the *divergence* of a vector field $\delta = \sum_i f_i \frac{\partial}{\partial x_i}$ is defined by $\mathrm{Div}\,\delta := \sum_i \frac{\partial f_i}{\partial x_i}$. This allows to define the following subspaces of $\mathrm{Vec}(\mathbb{A}^n)$,

$$\mathrm{Vec}^0(\mathbb{A}^n) := \{\delta \in \mathrm{Vec}(\mathbb{A}^n) \mid \mathrm{Div}\,\delta = 0\} \subset \mathrm{Vec}^c(\mathbb{A}^n) := \{\delta \in \mathrm{Vec}(\mathbb{A}^n) \mid \mathrm{Div}\,\delta \in K\},$$

which are Lie subalgebras, because $\mathrm{Div}[\delta, \eta] = \delta(\mathrm{Div}\,\eta) - \eta(\mathrm{Div}\,\delta)$. We have

$$\mathrm{Vec}^c(\mathbb{A}^n) = \mathrm{Vec}^0(\mathbb{A}^n) \oplus K\partial_E \text{ where } \partial_E := \sum_i x_i \frac{\partial}{\partial x_i} \text{ is the EULER } \textit{field}.$$

The aim of this note is to prove the following result about the automorphism groups of these Lie algebras.

Main Theorem. *There are canonical isomorphisms*

$$\mathrm{Aut}(\mathbb{A}^n) \xrightarrow{\sim} \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Vec}(\mathbb{A}^n)) \xrightarrow{\sim} \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Vec}^c(\mathbb{A}^n)) \xrightarrow{\sim} \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Vec}^0(\mathbb{A}^n)).$$

Remark 1.1. It is easy to see that the theorem holds for any field K of characteristic zero. In fact, all the homomorphisms are defined over $\mathbb{Q}$, and are equivariant with respect to the obvious actions of the Galois group $\Gamma = \mathrm{Gal}(\bar{K}/K)$.

As a consequence, we will get the next result (see Corollary 4.4) which goes back to KULIKOV [Kul92, Theorem 4].

Corollary. *If every injective endomorphism of the Lie algebra* $\mathrm{Vec}(\mathbb{A}^n)$ *is an automorphism, then the Jacobian Conjecture holds in dimension* n.

Remark 1.2. The Main Theorem has another interesting consequence. The group $\mathrm{Aut}(\mathbb{A}^n)$ is an *infinite-dimensional algebraic group* in the sense of SHAFAREVICH [Sha66, Sha81], shortly an *ind-group* (cf. [Kum02]), and its Lie algebra is canonically isomorphic to $\mathrm{Vec}^c(\mathbb{A}^n)$. It was recently shown by BELOV-KANEL and YU [BKY12] that every automorphism of $\mathrm{Aut}(\mathbb{A}^n)$ as an ind-group is inner. Using the Main Theorem above one can give a new proof of this and extend it to the closed subgroup $\mathrm{SAut}(\mathbb{A}^n) \subset \mathrm{Aut}(\mathbb{A}^n)$ of automorphisms with Jacobian determinant equal to 1. The details will appear in the forthcoming paper [Kra14] where we also show that the maps in the Main Theorem are isomorphisms of ind-groups.

We add here a lemma which will be used later on.

Lemma 1.3. $\mathrm{Vec}(\mathbb{A}^n)$ *and* $\mathrm{Vec}^0(\mathbb{A}^n)$ *are simple Lie algebras, and*

$$\mathrm{Vec}^0(\mathbb{A}^n) = [\mathrm{Vec}^c(\mathbb{A}^n), \mathrm{Vec}^c(\mathbb{A}^n)].$$

Proof. The formula $[\frac{\partial}{\partial x_j}, \sum_i f_i \frac{\partial}{\partial x_i}] = \sum_i \frac{\partial f_i}{\partial x_j}\frac{\partial}{\partial x_i}$ shows that every nonzero ideal $\mathfrak{a}$ of $\mathrm{Vec}(\mathbb{A}^n)$ contains a nonzero element from $\sum_i K\frac{\partial}{\partial x_i}$, and $[x_\ell \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_i}] = -\delta_{i\ell}\frac{\partial}{\partial x_j}$ implies that $\sum_i K\frac{\partial}{\partial x_i} \subseteq \mathfrak{a}$. Now we use $[f\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_i}] = -\frac{\partial f}{\partial x_i}\frac{\partial}{\partial x_j}$ to conclude that $\mathfrak{a} = \mathrm{Vec}(\mathbb{A}^n)$, hence $\mathrm{Vec}(\mathbb{A}^n)$ is simple. (See also [Jor78, Theorem on page 446].)

The second statement is proved in a similar way and can be found in [Sha81, Lemma 3], and from that the last claim follows immediately. □

2 Group actions and vector fields

If an algebraic group G acts on an affine variety X we obtain a canonical linear map $\mathrm{Lie}\, G \to \mathrm{Vec}(X)$ defined in the usual way (cf. [Kra11, II.4.4]). For every $A \in \mathrm{Lie}\, G$ the associated vector field ξ_A on X is defined by

$$(\xi_A)_x := d\mu_x(A) \text{ for } x \in X \tag{2.1}$$

where $\mu_x \colon G \to X$, $g \mapsto gx$, is the orbit map in $x \in X$. It is well-known that the linear map $A \mapsto \xi_A$ is a anti-homomorphism of Lie algebras, and that the kernel is equal to the Lie algebra of the kernel of the action $G \to \mathrm{Aut}(X)$. In particular, for any algebraic subgroup $G \subset \mathrm{Aut}(\mathbb{A}^n)$ we have a canonical injection $\mathrm{Lie}\, G \hookrightarrow \mathrm{Vec}(\mathbb{A}^n)$; we will denote the image by $L(G)$. Let us point out that a connected $G \subset \mathrm{Aut}(\mathbb{A}^n)$ is determined by $L(G)$, i.e., if $L(G) = L(H)$ for algebraic subgroups $G, H \subset \mathrm{Aut}(\mathbb{A}^n)$, then $G^0 = H^0$.

Recall that the vector field $\delta \in \mathrm{Vec}(\mathbb{A}^n)$ is called *locally nilpotent* if the action of δ on $K[x_1, \dots, x_n]$ is locally nilpotent, i.e., for any $f \in K[x_1, \dots, x_n]$ we have $\delta^m(f) = 0$ if m is large enough. Every such δ defines an action of the additive group K^+ on $\mathbb{A}^n$ such that $\delta = \xi_1$ where $1 \in K = \mathrm{Lie}\, K^+$ (see (2.1) above).

Lemma 2.1. *Let* $\mathbf{u} \subset \mathrm{Vec}(\mathbb{A}^n)$ *be a finite dimensional commutative Lie subalgebra consisting of locally nilpotent vector fields. Then there is a commutative unipotent algebraic subgroup* $U \subset \mathrm{Aut}(\mathbb{A}^n)$ *such that* $L(U) = \mathbf{u}$. *If* $\mathfrak{cent}_{\mathrm{Vec}(\mathbb{A}^n)}(\mathbf{u}) = \mathbf{u}$, *then* U *acts transitively on* $\mathbb{A}^n$.

Proof. It is clear that $\mathbf{u} = L(U)$ for a commutative unipotent subgroup $U \subset \mathrm{Aut}(\mathbb{A}^n)$. In fact, choose a basis $(\delta_1, \dots, \delta_m)$ if $\mathbf{u}$ and consider the corresponding actions $\rho_i \colon K^+ \to \mathrm{Aut}(\mathbb{A}^n)$. Since the associated vector fields δ_i commute, the same holds for the actions

ρ_i, so that we get an action of $(K^+)^m$. It follows that the image $U \subset \mathrm{Aut}(\mathbb{A}^n)$ is a commutative unipotent subgroup with $L(U) = \mathbf{u}$.

Assume that the action of U is not transitive. Then all orbits have dimension $< n$, because orbits of unipotent groups acting on affine varieties are closed (see [Bor91, Chap. I, Proposition 4.10]). But then there is a nonconstant U-invariant function $f \in K[x_1, \dots, x_n]$. This implies that for every $\delta \in \mathbf{u}$ the vector field $f\delta$ commutes with $\mathbf{u}$ and thus belongs to $\mathfrak{cent}_{\mathrm{Vec}(\mathbb{A}^n)}(\mathbf{u})$, contradicting the assumption. $\square$

Any $\delta \in \mathrm{Vec}(\mathbb{A}^n)$ acts on the functions $K[x_1, \dots, x_n]$ as a derivation, and on the Lie algebra $\mathrm{Vec}(\mathbb{A}^n)$ by the adjoint action, $\mathrm{ad}(\delta)\mu := [\delta, \mu] = \delta \circ \mu - \mu \circ \delta$. These two actions are related as shown in the following lemma whose proof is obvious.

Lemma 2.2. *Let $\delta, \mu \in \mathrm{Vec}(\mathbb{A}^n)$ be two commuting vector fields and $f \in K[x_1, \dots, x_n]$. Then*

$$\mathrm{ad}(\delta)(f\mu) = \delta(f)\mu.$$

In particular, if $\mathrm{ad}(\delta)$ is locally nilpotent on $\mathrm{Vec}(\mathbb{A}^n)$, then δ is locally nilpotent as a vector field.

3 Proof of the Main Theorem, part I

We first give a proof of the following result which goes back to KULIKOV [Kul92, Proof of Theorem 4]; see also [Bav13].

Theorem 3.1. *The canonical map $\mathrm{Ad}\colon \mathrm{Aut}(\mathbb{A}^n) \to \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Vec}(\mathbb{A}^n))$ is an isomorphism.*

Denote by $\mathrm{Aff}_n \subset \mathrm{Aut}(\mathbb{A}^n)$ the closed subgroup of affine transformations and by $S = (K^+)^n \subset \mathrm{Aff}_n$ the subgroup of translations. Then

$$L(\mathrm{Aff}_n) = \langle x_i\partial_{x_j}, \partial_{x_k} \mid 1 \leq i, j, k \leq n\rangle \supset L(S) = \langle \partial_{x_1}, \dots, \partial_{x_n}\rangle \tag{3.1}$$

where $\partial_{x_j} := \frac{\partial}{\partial x_j}$. Put $\mathfrak{aff}_n := \mathrm{Lie}\,\mathrm{Aff}_n$ and $\mathfrak{saff}_n := [\mathfrak{aff}_n, \mathfrak{aff}_n] = \mathrm{Lie}\,\mathrm{SAff}_n$ where $\mathrm{SAff}_n := (\mathrm{Aff}_n, \mathrm{Aff}_n) \subset \mathrm{Aff}_n$ is the commutator subgroup, i.e. the closed subgroup of those affine transformations $x \mapsto gx + b$ where $g \in \mathrm{SL}_n$. The next lemma is certainly known. For the convenience of the reader we indicated a short proof.

Lemma 3.2. *The canonical homomorphisms*

$$\mathrm{Aff}_n \xrightarrow[\simeq]{\mathrm{Ad}} \mathrm{Aut}_{\mathrm{Lie}}(\mathfrak{aff}_n) \xrightarrow[\simeq]{\mathrm{res}} \mathrm{Aut}_{\mathrm{Lie}}(\mathfrak{saff}_n)$$

are isomorphisms.

Proof. It is clear that the two homomorphisms

$$\mathrm{Ad}\colon \operatorname{Aff}_n \to \operatorname{Aut}_{\mathrm{Lie}}(\mathfrak{aff}_n) \text{ and } \mathrm{res}\colon \operatorname{Aut}_{\mathrm{Lie}}(\mathfrak{aff}_n) \to \operatorname{Aut}_{\mathrm{Lie}}(\mathfrak{saff}_n)$$

are both injective. Thus it suffices to show that the composition $\mathrm{res} \circ \mathrm{Ad}$ is surjective.

We write the elements of Aff_n in the form (v, g) with $v \in S = (K^+)^n$, $g \in \mathrm{GL}_n$ where $(v, g)x = gx + v$ for $x \in \mathbb{A}^n$. It follows that $(v, g)(w, h) = (v + gw, gh)$. Similarly, $(a, A) \in \mathfrak{aff}_n$ means that $a \in \mathfrak{s} := \operatorname{Lie} S = K^n$, $A \in \mathfrak{gl}_n$, and $(a, A)x = Ax + a$. For the adjoint representation of $g \in \mathrm{GL}_n$ and of $v \in S$ on $\mathfrak{aff}_n$ we find

$$\mathrm{Ad}(g)(a, A) = (ga, gAg^{-1}) \text{ and } \mathrm{Ad}(v)(a, A) = (a - Av, A), \tag{3.2}$$

and thus, for $(b, B) \in \mathfrak{aff}_n$,

$$\mathrm{ad}(B)(a, A) = (Ba, [B, A]) \text{ and } \mathrm{ad}(b)(a, A) = (a - Ab, A). \tag{3.3}$$

Now let θ be an automorphism of the Lie algebra $\mathfrak{saff}_n$. Then $\theta(\mathfrak{s}) = \mathfrak{s}$ since $\mathfrak{s}$ is the solvable radical of $\mathfrak{saff}_n$. Since $g := \theta|_{\mathfrak{s}} \in \mathrm{GL}_n$, we can replace θ by $\mathrm{Ad}(g^{-1}) \circ \theta$ and thus assume, by (3.2), that θ is the identity on $\mathfrak{s}$. This implies that $\theta(a, A) = (a + \ell(A), \bar{\theta}(A))$ where $\ell\colon \mathfrak{sl}_n \to \mathfrak{s}$ is a linear map and $\bar{\theta}\colon \mathfrak{sl}_n \xrightarrow{\sim} \mathfrak{sl}_n$ is a Lie algebra automorphism.

From (3.3) we get $\mathrm{ad}(b, B)(a, 0) = \mathrm{ad}(B)(a, 0) = (Ba, 0)$ for all $a \in \mathfrak{s}$, hence

$$\begin{aligned}(Ba, 0) = \theta(Ba, 0) = \theta(\mathrm{ad}(B)(a, 0)) = \\ = \mathrm{ad}(\theta(B))(a, 0) = \mathrm{ad}(\bar{\theta}(B))(a, 0) = (\bar{\theta}(B)a, 0).\end{aligned}$$

Thus $\bar{\theta}(B) = B$, i.e. $\theta(a, A) = (a + \ell(A), A)$. Now an easy calculation shows that $\ell([A, B]) = A\ell(B) - B\ell(A)$. This means that ℓ is a cocycle of $\mathfrak{sl}_n$. Since $\mathfrak{sl}_n$ is semisimple, ℓ is a coboundary and thus $\ell(A) = Av$ for a suitable $v \in K^n$. In view of (3.3) this implies that $\theta = \mathrm{Ad}(-v)$, and the claim follows. □

Proof of Theorem 3.1. It is clear that the homomorphism

$$\mathrm{Ad}\colon \operatorname{Aut}(\mathbb{A}^n) \to \operatorname{Aut}_{\mathrm{Lie}}(\operatorname{Vec}(\mathbb{A}^n))$$

is injective. So let $\theta \in \operatorname{Aut}_{\mathrm{Lie}}(\operatorname{Vec}(\mathbb{A}^n))$ be an arbitrary automorphism.

We have seen above that $L(S) = \langle \partial_{x_1}, \dots, \partial_{x_n} \rangle \subset \operatorname{Vec}(\mathbb{A}^n)$ where $S \subset \operatorname{Aff}_n$ is the subgroup of translations. Clearly, for every $\delta \in L(S)$ the adjoint action $\mathrm{ad}(\delta)$ on $\operatorname{Vec}(\mathbb{A}^n)$ is locally nilpotent, and the same holds for any element from $\mathbf{u} := \theta(L(S))$. It follows from Lemma 2.2 that $\mathbf{u}$ consists of locally nilpotent vector fields. Hence, by Lemma 2.1, $\mathbf{u} = L(U)$ for a commutative unipotent subgroup U of dimension n. Moreover, $\mathfrak{cent}_{\operatorname{Vec}(\mathbb{A}^n)}(L(S)) = L(S)$, and so $\mathfrak{cent}_{\operatorname{Vec}(\mathbb{A}^n)}(\mathbf{u}) = \mathbf{u}$ which implies, again by Lemma 2.1, that U acts transitively on $\mathbb{A}^n$. Thus every orbit map $U \to \mathbb{A}^n$ is an isomorphism. It follows that there is an automorphism $\varphi \in \operatorname{Aut}(\mathbb{A}^n)$ such that $\varphi U \varphi^{-1} = S$.

In fact, fix a group isomorphism $\psi\colon U \xrightarrow{\sim} S$ and take the orbit maps $\mu_S\colon S \xrightarrow{\sim} \mathbb{A}^n$ and $\mu_U\colon U \xrightarrow{\sim} \mathbb{A}^n$ at the origin $0 \in \mathbb{A}^n$. Then one easily sees that $\varphi := \mu_S \circ \psi \circ \mu_U^{-1}$ has the property that $\varphi \circ u \circ \varphi^{-1} = \psi(u)$ for all $u \in U$.

It follows that the automorphism $\theta' := \mathrm{Ad}(\varphi) \circ \theta \in \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Vec}(\mathbb{A}^n))$ sends $L(S)$ isomorphically onto itself. Now the relations $[\partial_{x_i}, x_j\partial_{x_k}] = \delta_{ij}\partial_{x_k}$ imply that $\theta'(L(\mathrm{Aff}_n)) = L(\mathrm{Aff}_n)$. By Lemma 3.2, there is an $\alpha \in \mathrm{Aff}_n$ such that $\mathrm{Ad}(\alpha) \circ \theta'$ is the identity on $L(\mathrm{Aff}_n)$. Hence, by the following lemma, $\mathrm{Ad}(\alpha) \circ \theta' = \mathrm{id}$, because $\mathrm{Ad}(\lambda E)$ acts by multiplication with λ on $L(S)$, and so $\theta = \mathrm{Ad}(\varphi^{-1} \circ \alpha^{-1})$. □

Lemma 3.3. *Let θ be an injective endomorphism of one of the Lie algebras $\mathrm{Vec}(\mathbb{A}^n)$, $\mathrm{Vec}^c(\mathbb{A}^n)$ or $\mathrm{Vec}^0(\mathbb{A}^n)$. If θ is the identity on $L(\mathrm{SL}_n)$, then $\theta = \mathrm{Ad}(\lambda E)$ for some $\lambda \in K^*$.*

Proof. We consider the action of GL_n on $\mathrm{Vec}(\mathbb{A}^n)$. Denote by $\mathrm{Vec}(\mathbb{A}^n)_d$ the homogeneous vector fields of degree d, i.e.

$$\mathrm{Vec}(\mathbb{A}^n)_d := \bigoplus_i K[x_1, \dots, x_n]_{d+1}\,\partial_{x_i} \simeq K[x_1, \dots, x_n]_{d+1} \otimes K^n.$$

Note that $\lambda E \in \mathrm{GL}_n$ acts by scalar multiplication with λ^{-d} on $\mathrm{Vec}(\mathbb{A}^n)_d$. We have split exact sequences of GL_n-modules

$$0 \longrightarrow \mathrm{Vec}^0(\mathbb{A}^n)_d \longrightarrow \mathrm{Vec}(\mathbb{A}^n)_d \xrightarrow{\mathrm{Div}} K[x_1, \dots, x_n]_d \longrightarrow 0 \tag{3.4}$$

where $K[x_1, \dots, x_n]_{-1} = (0)$. Moreover, the SL_n-modules $\mathrm{Vec}^0(\mathbb{A}^n)_d$ (for $d \geq -1$) and $K[x_1, \dots, x_n]_d$ (for $d \geq 0$) are simple and pairwise nonisomorphic (see PIERI's formula [Pro07, Chap. 9, section 10.2]). The splitting of (3.4) is given by $K[x_1, \dots, x_n]_d\partial_E \subset \mathrm{Vec}(\mathbb{A}^n)_d$ where $\partial_E = x_1\partial_{x_1} + \cdots + x_n\partial_{x_n}$ is the EULER field. In fact, the EULER field is fixed under GL_n and $\mathrm{Div}(f\partial_E) = (d+1)f$ for $f \in K[x_1, \dots, x_n]_d$.

Now let θ be an injective endomorphism of $\mathrm{Vec}(\mathbb{A}^n)$. If θ is the identity on $L(\mathrm{SL}_n)$, then θ is SL_n-equivariant and thus acts with a scalar λ_d on $\mathrm{Vec}^0(\mathbb{A}^n)_d$ and with a scalar μ_d on $K[x_1, \dots, x_n]_d\partial_E$, by SCHUR's Lemma. The relations

$$[x_j^{e+1}\partial_{x_i}, x_i^{d+1}\partial_{x_j}] = (d+1)x_i^d x_j^{e+1}\partial_{x_j} - (e+1)x_i^{d+1}x_j^e\partial_{x_i},\ i \neq j,$$

show that $\lambda_e\lambda_d = \lambda_{e+d}$, hence $\lambda_d = \lambda^d$ for $\lambda := \lambda_1$. The relations

$$[x_i^e\partial_E, x_i^d\partial_E] = (d-e)x_i^{e+d}\partial_E$$

show that $\mu_e\mu_d = \mu_{e+d}$ for $e \neq d$ which also implies that $\mu_d = \mu^d$ for $\mu := \mu_1$. Finally, from the relation $[\partial_{x_1}, x_2\partial_E] = x_2\partial_{x_1}$, we get $\lambda = \mu$, and so $\theta = \mathrm{Ad}(\lambda^{-1}\,\mathrm{id})$. This proves the claim for $\mathrm{Vec}(\mathbb{A}^n)$. The two other cases follow along the same lines. □

4 Étale Morphisms and Vector Fields

In the first section we defined the action of $\mathrm{Aut}(\mathbb{A}^n)$ on the vector fields $\mathrm{Vec}(\mathbb{A}^n)$ by the formula $\mathrm{Ad}(\varphi)\delta := \varphi^{*-1} \circ \delta \circ \varphi^*$. In more geometric terms, considering δ as a section of the tangent bundle $T\mathbb{A}^n = \mathbb{A}^n \times \mathbb{C}^n \to \mathbb{A}^n$, one defines the pull-back of δ by

$$\varphi^*(\delta) := (d\varphi)^{-1} \circ \delta \circ \varphi, \text{ i.e., } \varphi^*(\delta)_a = (d\varphi_a)^{-1}(\delta_{\varphi(a)}) \text{ for } a \in \mathbb{A}^n.$$

Clearly, $\varphi^*(\delta) = \mathrm{Ad}(\varphi^{-1})\delta$. However, the second formula above shows the well-known fact that the pull-back $\varphi^*(\delta)$ of a vector field δ is also defined for an étale morphism $\varphi\colon \mathbb{A}^n \to \mathbb{A}^n$. In the holomorphic setting this can be understood as lifting the corresponding integral curves.

Proposition 4.1. *Let $\varphi\colon \mathbb{A}^n \to \mathbb{A}^n$ be an étale morphism. For any vector field $\delta \in \mathrm{Vec}(\mathbb{A}^n)$ there is a vector field $\varphi^*(\delta) \in \mathrm{Vec}(\mathbb{A}^n)$ defined by $\varphi^*(\delta)_a := (d\varphi)_a^{-1}\,\delta_{\varphi(a)}$ for $a \in \mathbb{A}^n$. It is uniquely determined by*

$$\varphi^*(\delta)\varphi^*(f) = \varphi^*(\delta f) \text{ for } f \in K[x_1, \ldots, x_n]. \tag{4.1}$$

The map $\varphi^*\colon \mathrm{Vec}(\mathbb{A}^n) \to \mathrm{Vec}(\mathbb{A}^n)$ is an injective homomorphism of Lie algebras satisfying $\varphi^*(h\,\delta) = \varphi^*(h)\varphi^*(\delta)$ for $h \in K[x_1, \ldots, x_n]$. Moreover, $(\eta \circ \varphi)^* = \varphi^* \circ \eta^*$.

Proof. For a vector field $\delta\colon \mathbb{A}^n \to T\mathbb{A}^n$ and $a \in \mathbb{A}^n$ we have $(d\varphi \circ \delta)_a = d\varphi_a(\delta_a)$. Thus, the equation $(d\varphi)_a(\tilde\delta_a) = (\tilde\delta \circ \varphi)_a = \tilde\delta_{\varphi(a)}$ for the field $\tilde\delta$ has a unique solution, namely

$$\tilde\delta_a := (d\varphi_a)^{-1}(\delta_{\varphi(a)}),$$

which is well defined since $d\varphi_a$ is invertible. The Jacobian determinant $\det(\mathrm{Jac}(\varphi))$ is a nonzero constant, and so the inverse matrix $\mathrm{Jac}(\varphi)^{-1}$ has entries in $K[x_1, \ldots, x_n]$. Therefore, the vector field $\varphi^*(\delta) := \tilde\delta$ defined above is polynomial, and it satisfies the equation (4.1). This proves the first part of the proposition and shows that φ^* is injective. Using equation (4.1) we find

$$\varphi^*((\delta_1\delta_2)f) = \varphi^*(\delta_1(\delta_2 f)) = \varphi^*(\delta_1)\varphi^*(\delta_2 f) = (\varphi^*(\delta_1)\varphi^*(\delta_2))\varphi^*(f),$$

hence $\varphi^*([\delta_1, \delta_2]f) = [\varphi^*(\delta_1), \varphi^*(\delta_2)]\varphi^*(f)$, and so $\varphi^*([\delta_1, \delta_2]) = [\varphi^*(\delta_1), \varphi^*(\delta_2)]$. Moreover,

$$\varphi^*(h\delta)\varphi^*(f) = \varphi^*((h\delta)f) = \varphi^*(h)\varphi^*(\delta f) = \varphi^*(h)\varphi^*(\delta)\varphi^*(f),$$

hence $\varphi^*(h\delta) = \varphi^*(h)\varphi^*(\delta)$. This proves the second part of the proposition, and the last claim is obvious. $\square$

Remark 4.2. In the notation of the proposition above let $\varphi = (f_1, \dots, f_n)$. Then we get $\varphi^*(\delta x_i) = \varphi^*(\delta) f_i = \sum_j \frac{\partial f_i}{\partial x_j} \varphi^*(\delta) x_j$. Hence, for $\delta = \partial_{x_k}$, we obtain

$$\delta_{ik} = \varphi^*(\partial_{x_k}) f_i = \sum_j \frac{\partial f_i}{\partial x_j} \varphi^*(\partial_{x_k}) x_j.$$

This shows that the matrix $\Big(\varphi^*(\partial_{x_k}) x_j\Big)_{(j,k)}$ is invertible, $\Big(\varphi^*(\partial_{x_k}) x_j\Big)^{-1}_{(j,k)} = \mathrm{Jac}(\varphi)$, and that

$$\partial_{x_i} = \sum_j \frac{\partial f_i}{\partial x_j}\, \varphi^*(\partial_{x_j}). \tag{4.2}$$

Proposition 4.3. *Let $\varphi\colon \mathbb{A}^n \to \mathbb{A}^n$ be an étale morphism. Then the pull-back map*

$$\varphi^*\colon\ \mathrm{Vec}(\mathbb{A}^n) \to \mathrm{Vec}(\mathbb{A}^n)$$

is an isomorphism if and only if φ is an automorphism.

Proof. Assume that $\varphi^*\colon\ \mathrm{Vec}(\mathbb{A}^n) \to \mathrm{Vec}(\mathbb{A}^n)$ is an isomorphism. Since φ is étale, the comorphism $\varphi^*\colon K[x_1, \dots, x_n] \to K[x_1, \dots, x_n]$ is injective, and we only have to show that it is surjective. Proposition 4.1 implies that $\varphi^*(\mathrm{Vec}(\mathbb{A}^n)) = \sum_i \varphi^*(K[x_1, \dots, x_n])\varphi^*(\partial_{x_i})$, and from equation (4.2) above, we get

$$\mathrm{Vec}(\mathbb{A}^n) = \oplus_i K[x_1, \dots, x_n]\partial_{x_i} = \oplus_i K[x_1, \dots, x_n]\varphi^*(\partial_{x_i}).$$

Hence $\varphi^*(\mathrm{Vec}(\mathbb{A}^n)) = \mathrm{Vec}(\mathbb{A}^n)$ if and only if $\varphi^*(K[x_1, \dots, x_n]) = K[x_1, \dots, x_n]$. $\square$

As a corollary of the two propositions above, we get the following result which is due to KULIKOV [Kul92, Theorem 4].

Corollary 4.4. *If every injective endomorphism of the Lie algebra $\mathrm{Vec}(\mathbb{A}^n)$ is an automorphism, then the Jacobian Conjecture holds in dimension n.*

Remark 4.5. The result of KULIKOV is stronger. He proves that every injective endomorphism of $\mathrm{Vec}(\mathbb{A}^n)$ is induced by an étale map φ which implies also the converse of the statement above: *If the Jacobian Conjecture holds in dimension n, then every injective endomorphism of $\mathrm{Vec}(\mathbb{A}^n)$ is an automorphism.*

We finish this section by showing that if the divergence of a vector field is a constant, then it is invariant under an étale morphism. More generally, we have the following result.

Proposition 4.6. *Let $\varphi\colon \mathbb{A}^n \to \mathbb{A}^n$ be an étale morphism, and let δ be a vector field. Then $\mathrm{Div}\, \varphi^*(\delta) = \varphi^*(\mathrm{Div}\, \delta)$. In particular, $\delta \in \mathrm{Vec}^c(\mathbb{A}^n)$ if and only if $\varphi^*(\delta) \in \mathrm{Vec}^c(\mathbb{A}^n)$, and in this case we have $\mathrm{Div}\, \varphi^*(\delta) = \mathrm{Div}\, \delta$.*

Proof. Set $\varphi = (f_1, \ldots, f_n)$, $\delta = \sum_j h_j \partial_{x_j}$ and $\varphi^*(\delta) = \sum_j \tilde{h}_j \partial_{x_j}$. Then, by (4.1),

$$h_k(f_1, \ldots, f_n) = \sum_i \tilde{h}_i \frac{\partial f_k}{\partial x_i} \text{ for } k = 1, \ldots, n.$$

Applying $\frac{\partial}{\partial x_j}$ to the left hand side we get the matrix

$$\left(\sum_i \frac{\partial h_k}{\partial x_i}(f_1, \ldots, f_n) \frac{\partial f_i}{\partial x_j} \right)_{(k,j)} = H(f_1, \ldots, f_n) \cdot \operatorname{Jac}(\varphi)$$

where $H := \operatorname{Jac}(h_1, \ldots, h_n)$. On the right hand side, we obtain similarly

$$\left(\sum_i \frac{\partial \tilde{h}_i}{\partial x_j} \frac{\partial f_k}{\partial x_i} + \sum_i \tilde{h}_i \frac{\partial^2 f_k}{\partial x_i \partial x_j} \right)_{(k,j)} = \tilde{H} \cdot \operatorname{Jac}(\varphi) + \sum_i \tilde{h}_i \frac{\partial}{\partial x_i} \operatorname{Jac}(\varphi)$$

Multiplying this matrix equation from the right with $\operatorname{Jac}(\varphi)^{-1}$ we finally get

$$H(f_1, \ldots, f_n) = \tilde{H} + \sum_i \tilde{h}_i \frac{\partial}{\partial x_i} \operatorname{Jac}(\varphi) \cdot \operatorname{Jac}(\varphi)^{-1}$$

Now we take on both sides the traces. Using Lemma 4.7 below and the obvious equalities $\operatorname{Div} \delta = \operatorname{tr} H$ and $\operatorname{Div} \tilde{\delta} = \operatorname{tr} \tilde{H}$, we finally get

$$\operatorname{Div} \tilde{\delta} = (\operatorname{Div} \delta)(f_1, \ldots, f_n) = \varphi^*(\operatorname{Div} \delta).$$

The claim follows. □

Lemma 4.7. *Let A be an $n \times n$ matrix whose entries $a_{ij}(t)$ are polynomials in t. Then*

$$\operatorname{tr} \left(\frac{d}{dt} A \cdot \operatorname{Adj}(A) \right) = \frac{d}{dt} \det A$$

where $\operatorname{Adj}(A)$ is the adjoint matrix of A.

The proof is a nice exercise in linear algebra which we leave to the reader! It holds for rational entries $a_{ij}(t)$ over any field K, and in case $K = \mathbb{R}$ or $\mathbb{C}$ also for differentiable entries $a_{ij}(t)$.

5 Proof of the Main Theorem, part II

We have seen that the canonical map $\operatorname{Ad}\colon \operatorname{Aut}(\mathbb{A}^n) \to \operatorname{Aut}_{\text{Lie}}(\operatorname{Vec}(\mathbb{A}^n))$ is an isomorphism (Theorem 3.1). It follows from Proposition 4.6 that every automorphism of $\operatorname{Vec}(\mathbb{A}^n)$ induces an automorphism of $\operatorname{Vec}^c(\mathbb{A}^n)$. Moreover, since

$$\operatorname{Vec}^0(\mathbb{A}^n) = [\operatorname{Vec}^c(\mathbb{A}^n), \operatorname{Vec}^c(\mathbb{A}^n)]$$

(Lemma 1.3), we get a canonical map $\operatorname{Aut}_{\text{Lie}}(\operatorname{Vec}^c(\mathbb{A}^n)) \to \operatorname{Aut}_{\text{Lie}}(\operatorname{Vec}^0(\mathbb{A}^n))$ which is easily seen to be injective. Thus the main theorem follows from the next result.

Theorem 5.1. *The canonical map* $\mathrm{Ad}\colon \mathrm{Aut}(\mathbb{A}^n) \to \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Vec}^0(\mathbb{A}^n))$ *is an isomorphism.*

The proof needs some preparation. The next proposition is a reformulation of some results from [Now86] and [LD12]. For the convenience of the reader we will give a short proof.

Proposition 5.2. *Let* $\delta_1,\ldots,\delta_n \in \mathrm{Vec}(\mathbb{A}^n)$ *be pairwise commuting and* K*-linearly independent vector fields. Then the following statements are equivalent.*

(i) *There is an étale morphism* $\varphi\colon \mathbb{A}^n \to \mathbb{A}^n$ *such that* $\varphi^*(\partial_{x_i}) = \delta_i$ *for all* i*;*

(ii) $\mathrm{Vec}(\mathbb{A}^n) = \bigoplus_i K[x_1,\ldots,x_n]\delta_i$*;*

(iii) *There exist* $f_1,\ldots,f_n \in K[x_1,\ldots,x_n]$ *such that* $\delta_i(f_j) = \delta_{ij}$*;*

(iv) $\delta_1,\ldots,\delta_n$ *do not have a common* DARBOUX *polynomial.*

Recall that a common DARBOUX polynomial of the δ_i is a nonconstant polynomial $f \in K[x_1,\ldots,x_n]$ such that $\delta_i(f) = h_i f$ for some $h_i \in K[x_1,\ldots,x_n]$, $i = 1,\ldots,n$.

Proof. (a) It follows from Remark 4.2 that (i) implies (ii) and (iii). Clearly, (ii) implies (iv) since a common DARBOUX polynomial for the δ_i is also a common DARBOUX polynomial for the ∂_{x_i} which does not exist.

(b) We now show that (ii) implies (i), hence (iii), using the following well-known fact. If $h_1,\ldots,h_n \in K[x_1,\ldots,x_n]$ satisfy the conditions $\frac{\partial h_i}{\partial x_j} = \frac{\partial h_j}{\partial x_i}$ for all i,j, then there is an $f \in K[x_1,\ldots,x_n]$ such that $h_i = \frac{\partial f}{\partial x_i}$ for all i.

By (ii) we have $\partial_{x_i} = \sum_k h_{ik}\delta_k$ for $i = 1,\ldots,n$. We claim that $\frac{\partial h_{ik}}{\partial x_j} = \frac{\partial h_{jk}}{\partial x_i}$ for all i,j,k. In fact,

$$
\begin{aligned}
0 &= \partial_{x_i}\partial_{x_j} - \partial_{x_j}\partial_{x_i} = \partial_{x_i}\sum_k h_{jk}\delta_k - \partial_{x_j}\sum_k h_{ik}\delta_k = \\
&= \sum_k \frac{\partial h_{jk}}{\partial x_i}\delta_k + \sum_k h_{jk}\partial_{x_i}\delta_k - \sum_k \frac{\partial h_{ik}}{\partial x_j}\delta_k - \sum_k h_{ik}\partial_{x_j}\delta_k = \\
&= \sum_k \left(\frac{\partial h_{jk}}{\partial x_i} - \frac{\partial h_{ik}}{\partial x_j}\right)\delta_k + \left(\sum_{k,\ell} h_{jk}h_{i\ell}\delta_\ell\delta_k - \sum_{k,\ell} h_{ik}h_{j\ell}\delta_\ell\delta_k\right) = \\
&= \sum_k \left(\frac{\partial h_{jk}}{\partial x_i} - \frac{\partial h_{ik}}{\partial x_j}\right)\delta_k + \sum_{k,\ell} h_{ik}h_{j\ell}[\delta_k,\delta_\ell] = \\
&= \sum_k \left(\frac{\partial h_{jk}}{\partial x_i} - \frac{\partial h_{ik}}{\partial x_j}\right)\delta_k.
\end{aligned}
$$

Hence $h_{ik} = \frac{\partial f_k}{\partial x_i}$ for suitable $f_1, \dots, f_n \in K[x_1, \dots, x_n]$. It is clear that the matrix (h_{ik}) is invertible. This implies that the morphism $\varphi := (f_1, \dots, f_n)\colon \mathbb{A}^n \to \mathbb{A}^n$ is étale, and $\partial_{x_i} = \sum_k \frac{\partial f_k}{\partial x_i}\delta_k$, hence $\delta_k = \varphi^*(\partial_{x_k})$, by equation (4.2) in Remark 4.2.

(c) Assume that (iii) holds. Setting $\delta_i = \sum_k h_{ik}\partial_{x_k}$ and applying both sides to f_j, we see that the matrix $(h_{ik}) \in \mathrm{M}_n(K[x_1, \dots, x_n])$ is invertible, hence (ii). Thus the first three statements of the proposition are equivalent, and they imply (iv).

(d) Finally, assume that (iv) holds. Put $\delta_i = \sum_k h_{ik}\partial_{x_k}$. Since $[\delta_i, \delta_j] = 0$ we get $\delta_i(h_{jk}) = \delta_j(h_{ik})$ for all i, j, k. Now an easy calculation shows that $\delta_k(\det(h_{ij})) = \mathrm{Div}(\delta_k)\det(h_{ij})$, and so $\det(h_{ij}) \in K$. If $\det(h_{ij}) \neq 0$, then (ii) follows.

If $\det(h_{ij}) = 0$, then $\mathrm{rank}(\sum_{i=1}^n K[x_1, \dots, x_n]\delta_i) = r < n$, and we can assume that $\mathrm{rank}(\sum_{i=1}^r K[x_1, \dots, x_n]\delta_i) = r$. Choose a nontrivial relation $\sum_{i=1}^{r+1} f_i\delta_i = 0$ where $\gcd(f_1, \dots, f_{r+1}) = 1$. Since $0 = \delta_j(\sum_{i=1}^{r+1} f_i\delta_i) = \sum_{i=1}^{r+1}\delta_j(f_i)\delta_i$ for any j we see that $\delta_j(f_i) \in K[x_1, \dots, x_n]f_i$, and since the δ_j are K-linearly independent, at least one of the f_i is not a constant, hence a common DARBOUX polynomial, contradicting (iv). $\square$

The second main ingredient for the proof is the following result.

Lemma 5.3. *Let $\delta_1, \delta_2 \in \mathrm{Vec}^0(\mathbb{A}^n)$ be commuting vector fields. Assume that*

(a) δ_1 and δ_2 have a common DARBOUX *polynomial f where $\delta_i f \neq 0$, $i = 1, 2$.*

(b) Each δ_i acts locally nilpotently on $\mathrm{Vec}^0(\mathbb{A}^n)$.

Then $K[x_1, \dots, x_n]\delta_1 + K[x_1, \dots, x_n]\delta_2 \subseteq \mathrm{Vec}(\mathbb{A}^n)$ is a $K[x_1, \dots, x_n]$-submodule of rank ≤ 1.

Proof. We will show that there are nonzero polynomials p_1, p_2 such that $p_1\delta_1 = p_2\delta_2$. We have $\delta_i(f) = h_i f$ where $h_1, h_2 \neq 0$. Since δ_1 and δ_2 commute we get $\delta_1(h_2 f) = \delta_2(h_1 f)$, and so $\delta_1 h_2 = \delta_2 h_1$. Using the formula $\mathrm{Div}(g\delta) = \delta g + g\,\mathrm{Div}(\delta)$, this implies that $\delta := h_1\delta_2 - h_2\delta_1 \in \mathrm{Vec}^0(\mathbb{A}^n)$. Moreover, $\delta f = 0$, and so $f\delta \in \mathrm{Vec}^0(\mathbb{A}^n)$. Since

$$[\delta_1, \xi] = [\delta_1, h_1\delta_2] - [\delta_1, h_2\delta_1] = (\delta_1 h_1)\delta_2 - (\delta_1 h_2)\delta_1,$$

we get $(\mathrm{ad}\,\delta_1)^k\delta = \delta_1^k(h_1)\delta_2 - \delta_1^k(h_2)\delta_1$ and $(\mathrm{ad}\,\delta_1)^k(f\delta) = \delta_1^k(fh_1)\delta_2 - \delta_1^k(fh_2)\delta_1$. Now, by assumption (b), there is a $k > 0$ such that $(\mathrm{ad}\,\delta_1)^k\delta = (\mathrm{ad}\,\delta_1)^k(f\delta) = 0$, hence

$$\delta_1^k(h_1)\delta_2 = \delta_1^k(h_2)\delta_1 \text{ and } \delta_1^k(fh_1)\delta_2 = \delta_1^k(fh_2)\delta_1.$$

Thus the claim follows except if $\delta_1^k h_1 = \delta_1^k h_2 = \delta_1^k(fh_1) = \delta_1^k(fh_2) = 0$. We will show that this leads to a contradiction. Since $\delta_1 f = h_1 f$, we get $\delta_1^{k+1} f = 0$. Choose r, s minimal with $\delta_1^r h_1 = 0$ and $\delta_1^s f = 0$. By assumption, $r, s \geq 1$, and we get $\delta_1^{r+s-2}(h_1 f) = \delta_1^{r-1}h_1 \cdot \delta_1^{s-1} f \neq 0$. On the other hand, we have $\delta_1^{s-1}(h_1 f) = \delta_1^s f = 0$, and we end up with a contradiction, because $s - 1 \leq r + s - 2$. $\square$

Now we can prove the Theorem.

Proof of Theorem 5.1. The case $n = 1$ is handled in Lemma 3.2, so we can assume that $n \geq 2$. Let θ be an automorphism of $\mathrm{Vec}^0(\mathbb{A}^n)$ as a Lie algebra, and put $\delta_i := \theta(\partial_{x_i})$. Then the vector fields $\delta_1, \ldots, \delta_n$ are pairwise commuting and K-linearly independent. Since ∂_{x_i} acts locally nilpotently on $\mathrm{Vec}^0(\mathbb{A}^n)$ the same holds for δ_i. Moreover, the centralizer of the δ_i in $\mathrm{Vec}^0(\mathbb{A}^n)$ is the linear span of the δ_i, i.e. $[\delta, \delta_i] = 0$ for all i implies that $\delta \in \bigoplus_i K\delta_i$. In the following we will use vector fields with rational coefficients:

$$\mathrm{Vec}^{rat}(\mathbb{A}^n) := K(x_1, \ldots, x_n) \otimes_{K[x_1,\ldots,x_n]} \mathrm{Vec}(\mathbb{A}^n) = \bigoplus_{i=1}^{n} K(x_1, \ldots, x_n)\partial_{x_i}.$$

(1) We first claim that the δ_i do not have a common DARBOUX polynomial. So assume that there exists a nonconstant $f \in K[x_1, \ldots, x_n]$ such that $\delta_i f = h_i f$ for all i and suitable $h_i \in K[x_1, \ldots, x_n]$.

First assume that $h_1 = 0$, i.e. $\delta_1 f = 0$. Then $f\delta_1 \in \mathrm{Vec}^0(\mathbb{A}^n)$, and for any $h \in K[x_1, \ldots, x_n]$ and every i we have $[\delta_i, hf\delta_1] = \delta_i(hf)\delta_1 = (\delta_i(h) + hh_i)f\delta_1$, and so

$$(\mathrm{ad}\,\delta_i)^k(K[x_1, \ldots, x_n]f\delta_1) \subseteq K[x_1, \ldots, x_n]f\delta_1 \text{ for all } k \geq 0. \tag{5.1}$$

Set $\eta := \theta^{-1}(f\delta_1)$. Then there are $k_1, \ldots, k_n \in \mathbb{N}$ such that

$$\eta_0 := (\mathrm{ad}\,\partial_{x_1})^{k_1}(\mathrm{ad}\,\partial_{x_2})^{k_2} \cdots (\mathrm{ad}\,\partial_{x_n})^{k_n}\eta \in K\partial_{x_1} \oplus \cdots \oplus K\partial_{x_n} \setminus \{0\}.$$

Hence, $\theta(\eta_0) = (\mathrm{ad}\,\delta_1)^{k_1}(\mathrm{ad}\,\delta_2)^{k_2} \cdots (\mathrm{ad}\,\delta_n)^{k_n}(f\delta_1) \in K\delta_1 \oplus \cdots \oplus K\delta_n \setminus \{0\}$ which contradicts (5.1), because $f \notin K$.

We are left with the case where all $h_i \neq 0$. Then, Lemma 5.3 above implies that $\sum_i K[x_1, \ldots, x_n]\,\delta_i \subseteq \mathrm{Vec}(\mathbb{A}^n)$ has rank 1, i.e., there exist a $\delta \in \mathrm{Vec}(\mathbb{A}^n)$ and nonzero rational functions $r_i \in K(x_1, \ldots, x_n)$ such that $\delta_i = r_i\delta$ for $i = 1, \ldots, n$. We can assume that δ is minimal, i.e., that δ is not of the form $q\,\delta'$ with a nonconstant polynomial q. For every μ commuting with δ_i, we get $0 = [\mu, \delta_i] = [\mu, r_i\delta] = \mu(r_i)\delta + r_i[\mu, \delta]$, hence $[\mu, \delta] \in K(x_1, \ldots, x_n)\delta$. It is easy to see that

$$L := \{\xi \in \mathrm{Vec}(\mathbb{A}^n) \mid [\xi, \delta] \in K(x_1, \ldots, x_n)\delta\}$$

is a Lie subalgebra of $\mathrm{Vec}(\mathbb{A}^n)$ which contains all elements commuting with one of the δ_i. Since $\mathrm{Vec}^0(\mathbb{A}^n)$ is generated, as a Lie algebra, by elements commuting with one of the ∂_{x_i}, we see that $\theta(\mathrm{Vec}^0(\mathbb{A}^n)) = \mathrm{Vec}^0(\mathbb{A}^n)$ is generated by the elements commuting with one of the δ_i. Thus $\mathrm{Vec}^0(\mathbb{A}^n) \subseteq L$, and so $[\mathrm{Vec}^0(\mathbb{A}^n), \delta] \subseteq K(x_1, \ldots, x_n)\delta$. For $\delta = \sum_i p_i\partial_{x_i}$ we get $[\partial_{x_k}, \delta] = \sum_i \frac{\partial p_i}{\partial x_k}\partial_{x_i} = s\delta$ for some $s \in K(x_1, \ldots, x_n)$, hence $\frac{\partial p_i}{\partial x_k}p_j = \frac{\partial p_j}{\partial x_k}p_i$ for all pairs i, j. This implies that $\frac{\partial}{\partial x_k}\frac{p_j}{p_i} = 0$ in case $p_i \neq 0$, i.e. $\frac{p_j}{p_i}$ does not depend on x_k. Since this holds for all k, we conclude that $p_j = c_j p_i$ for some $c_j \in K$, hence $\delta = \sum_j c_j \partial_{x_j}$, because δ is minimal. In particular, $[\partial_{x_k}, \delta] = 0$ for all k. Now we get $[x_\ell \partial_{x_k}, \delta] = -c_\ell \partial_{x_k} \in K(x_1, \ldots, x_n)\delta$ for all k, ℓ which implies $\delta = 0$, hence a contradiction.

(2) Now we use the implication (vi) $\Rightarrow$ (i) of Proposition 5.2 to see that there is an étale morphism $\varphi\colon \mathbb{A}^n \to \mathbb{A}^n$ with $\delta_i = \varphi^*(\partial_{x_i})$ for all i. Then the composition $\theta' := \theta^{-1} \circ \varphi^*\colon \operatorname{Vec}^0(\mathbb{A}^n) \to \operatorname{Vec}^0(\mathbb{A}^n)$ is an injective homomorphism of Lie algebras (Proposition 4.1) and $\theta'(\partial_{x_i}) = \partial_{x_i}$. Hence, Lemma 5.4 below implies that $\theta' = \operatorname{Ad}(s) = (s^{-1})^*$ where $s \in \operatorname{Aut}(\mathbb{A}^n)$ is a translation, hence $\theta = (\varphi \circ s)^*$. Now Proposition 4.3 implies that $\psi := \varphi \circ s$ is an automorphism of $\mathbb{A}^n$, and so $\theta = \operatorname{Ad}(\psi^{-1})$ as claimed. □

Lemma 5.4. *Let θ be an injective endomorphism of $\operatorname{Vec}^0(\mathbb{A}^n)$ such that $\theta(\partial_{x_i}) = \partial_{x_i}$ for all i. Then $\theta = \operatorname{Ad}(s)$ where $s\colon \mathbb{A}^n \xrightarrow{\sim} \mathbb{A}^n$ is a translation. In particular, θ is an automorphism.*

Proof. We know that $\sum_i K\partial_{x_i} = L(S)$ where $S \subset \operatorname{Aff}_n$ are the translations. Moreover, $L(\operatorname{Aff}_n)$ is the normalizer of $L(S)$ in the Lie algebra $\operatorname{Vec}(\mathbb{A}^n)$. Hence $\theta(L(\operatorname{SAff}_n)) = L(\operatorname{SAff}_n)$, and so there is an affine transformation g such that $\operatorname{Ad}(g)|_{L(\operatorname{SAff}_n)} = \theta|_{L(\operatorname{SAff}_n)}$, by Lemma 3.2. Since $\operatorname{Ad}(g)$ is the identity on $L(S)$ we see that g is a translation. It follows that $\operatorname{Ad}(g^{-1}) \circ \theta$ is the identity on $L(\operatorname{SL}_n)$, hence $\operatorname{Ad}(g^{-1}) \circ \theta = \operatorname{Ad}(\lambda E)$ for some $\lambda \in K^*$, by Lemma 3.3. But $\lambda = 1$, because θ is the identity on $L(S)$, and so $\theta = \operatorname{Ad}(g)$. □

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LIE SUBALGEBRAS OF VECTOR FIELDS ON AFFINE 2-SPACE AND THE JACOBIAN CONJECTURE

ANDRIY REGETA

Abstract. We study Lie subalgebras L of the vector fields $\mathrm{Vec}^c(\mathbb{A}^2)$ of affine 2-space $\mathbb{A}^2$ of constant divergence, and we classify those L which are isomorphic to the Lie algebra $\mathfrak{aff}_2$ of the group $\mathrm{Aff}_2(K)$ of affine transformations of $\mathbb{A}^2$. We then show that the following statements are equivalent:

(a) The Jacobian Conjecture holds in dimension 2;
(b) All Lie subalgebras $L \subset \mathrm{Vec}^c(\mathbb{A}^2)$ isomorphic to $\mathfrak{aff}_2$ are conjugate under $\mathrm{Aut}(\mathbb{A}^2)$;
(c) All Lie subalgebras $L \subset \mathrm{Vec}^c(\mathbb{A}^2)$ isomorphic to $\mathfrak{aff}_2$ are algebraic.

Finally, we use these results to show that the automorphism groups of the Lie algebras $\mathrm{Vec}(\mathbb{A}^2)$, $\mathrm{Vec}^0(\mathbb{A}^2)$ and $\mathrm{Vec}^c(\mathbb{A}^2)$ are all isomorphic to $\mathrm{Aut}(\mathbb{A}^2)$.

1. Introduction

Let K be an algebraically closed field of characteristic zero. It is a well-known consequence of the amalgamated product structure of $\mathrm{Aut}(\mathbb{A}^2)$ that every reductive subgroup $G \subset \mathrm{Aut}(\mathbb{A}^2)$ is conjugate to a subgroup of $\mathrm{GL}_2(\mathbb{C}) \subset \mathrm{Aut}(\mathbb{A}^2)$, i.e. there is a $\psi \in \mathrm{Aut}(\mathbb{A}^2)$ such that $\psi G \psi^{-1} \subset \mathrm{GL}_2(\mathbb{C})$ ([Kam79], cf. [Kra96]). The "Linearization Problem" asks whether the same holds for $\mathrm{Aut}(\mathbb{A}^n)$. It was shown by Schwarz in [Sch89] that this is not the case in dimensions $n \geq 4$ (cf. [Kno91]).

In this paper we consider the analogue of the Linearization Problem for Lie algebras. It is known that the Lie algebra $\mathrm{Lie}(\mathrm{Aut}(\mathbb{A}^2))$ of the ind-group $\mathrm{Aut}(\mathbb{A}^2)$ is canonically isomorphic to the Lie algebra $\mathrm{Vec}^c(\mathbb{A}^2)$ of vector fields of constant divergence ([Sha66, Sha81], cf. [Kum02]). We will see that the Lie subalgebra

$$L := K(x^2\partial_x - 2xy\partial_y) \oplus K(x\partial_x - y\partial_y) \oplus K\partial_x \subset \mathrm{Vec}^c(\mathbb{A}^2)$$

where $\partial_x := \frac{\partial}{\partial x}$ and $\partial_y := \frac{\partial}{\partial y}$, is isomorphic to $\mathfrak{sl}_2$, but not conjugate to the standard $\mathfrak{sl}_2 \subset \mathrm{Vec}^c(\mathbb{A}^2)$ under $\mathrm{Aut}(\mathbb{A}^2)$ (Remark 4.3). However, for some other Lie subalgebras of $\mathrm{Vec}^c(\mathbb{A}^2)$, the situation is different. Let $\mathrm{Aff}_2(K) \subset \mathrm{Aut}(\mathbb{A}^2)$ be the group of affine transformations and $\mathrm{SAff}_2(K) \subset \mathrm{Aff}_2(K)$ the subgroup of affine transformations with determinant equal to 1, and denote by $\mathfrak{aff}_2$, respectively $\mathfrak{saff}_2$ their Lie algebras which we consider as subalgebras of $\mathrm{Vec}^c(\mathbb{A}^2)$. The first result we prove is the following (see Proposition 3.9). For $f \in K[x,y]$ we set $D_f := f_x\partial_y - f_y\partial_x \in \mathrm{Vec}^c(\mathbb{A}^2)$.

Theorem A. *Let $L \subset \mathrm{Vec}^c(\mathbb{A}^2)$ be a Lie subalgebra isomorphic to $\mathfrak{aff}_2$. Then there is an étale map $\varphi\colon \mathbb{A}^2 \to \mathbb{A}^2$ such that $L = \varphi^*(\mathfrak{aff}_2)$. More precisely, if (D_f, D_g) is a basis of the solvable radical of $[L,L]$, then*

$$L = \langle D_f, D_g, D_{f^2}, D_{g^2}, fD_g, gD_f \rangle,$$

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and one can take $\varphi = (f, g)$.

The analogous statements hold for Lie subalgebras isomorphic to $\mathfrak{saff}_2$. As a consequence of this classification we obtain the next result (see Theorem 4.1 and Corollary 4.4). Recall that a Lie subalgebra of $\mathrm{Vec}(\mathbb{A}^2)$ is *algebraic* if it acts locally finitely on $\mathrm{Vec}(\mathbb{A}^2)$.

Theorem B. *The following statements are equivalent:*

(i) *The Jacobian Conjecture holds in dimension 2;*
(ii) *All Lie subalgebras $L \subset \mathrm{Vec}^c(\mathbb{A}^2)$ isomorphic to $\mathfrak{aff}_2$ are conjugate under $\mathrm{Aut}(\mathbb{A}^2)$;*
(iii) *All Lie subalgebras $L \subset \mathrm{Vec}^c(\mathbb{A}^2)$ isomorphic to $\mathfrak{saff}_2$ are conjugate under $\mathrm{Aut}(\mathbb{A}^2)$;*
(iv) *All Lie subalgebras $L \subset \mathrm{Vec}^c(\mathbb{A}^2)$ isomorphic to $\mathfrak{aff}_2$ are algebraic;*
(v) *All Lie subalgebras $L \subset \mathrm{Vec}^c(\mathbb{A}^2)$ isomorphic to $\mathfrak{saff}_2$ are algebraic.*

Finally, as a consequence of the theorem above, we can determine the automorphism groups of the Lie algebras of vector fields (Theorem 4.5).

Theorem C. *There are canonical isomorphisms*

$$\mathrm{Aut}(\mathbb{A}^2) \xrightarrow{\sim} \mathrm{Aut}_{LA}(\mathrm{Vec}(\mathbb{A}^2)) \xrightarrow{\sim} \mathrm{Aut}_{LA}(\mathrm{Vec}^c(\mathbb{A}^2)) \xrightarrow{\sim} \mathrm{Aut}_{LA}(\mathrm{Vec}^0(\mathbb{A}^2)).$$

(Here $\mathrm{Vec}^0(\mathbb{A}^2)$ denotes the vector fields with zero divergence, see section 2).

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2. The Poisson algebra

Definitions. Let K be an algebraically closed field of characteristic zero and let P be the *Poisson algebra*, i.e., the Lie algebra with underlying vector space $K[x, y]$ and with Lie bracket $\{f, g\} := f_x g_y - f_y g_x$ for $f, g \in P$. If $\mathrm{Jac}(f, g)$ denotes the *Jacobian matrix* and $j(f, g)$ the *Jacobian determinant*,

$$\mathrm{Jac}(f, g) := \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}, \quad j(f, g) := \det \mathrm{Jac}(f, g),$$

then $\{f, g\} = j(f, g)$. Denote by $\mathrm{Vec}(\mathbb{A}^2)$ the polynomial vector fields on affine 2-space $\mathbb{A}^2 = K^2$, i.e. the derivations of $K[x, y]$:

$$\mathrm{Vec}(\mathbb{A}^2) := \{p\partial_x + q\partial_y \mid p, q \in K[x, y]\} = \mathrm{Der}(K[x, y]).$$

There is a canonical homomorphism of Lie algebras

$$\mu \colon P \to \mathrm{Vec}(\mathbb{A}^2), \ h \mapsto D_h := h_x\partial_y - h_y\partial_x,$$

with kernel $\ker \mu = K$.

The next lemma lists some properties of the Lie algebra P. These results are essentially known, see e.g. [NN88]. If L is any Lie algebra and $X \subset L$ a subset, we define the *centralizer* of X by

$$\mathfrak{cent}_L(X) := \{z \in L \mid [z, x] = 0 \text{ for all } x \in X\},$$

and we shortly write $\mathfrak{cent}(L)$ for the *center* of L.

Lemma 2.1. (a) *The center of P consists exactly of the constants $K \subset P$.*
(b) *If $f, g \in P$ are such that $\{f, g\} = 0$, then $f, g \in K[h]$ for some $h \in K[x, y]$.*

(c) *If $f, g \in P$ such that $\{f, g\} \neq 0$, then f, g are algebraically independent in $K[x, y]$, and $\mathfrak{cent}_P(f) \cap \mathfrak{cent}_P(g) = K$.*
(d) *P is generated, as a Lie algebra, by $\{x, x^3, y^2\}$.*

Proof. (a) is easy and left to the reader.

(b) Consider the morphism $\varphi = (f, g)\colon \mathbb{A}^2 \to \mathbb{A}^2$. Then $C := \overline{\varphi(\mathbb{A}^2)} \subset \mathbb{A}^2$ is an irreducible rational curve, and we have a factorization

$$\varphi\colon \mathbb{A}^2 \xrightarrow{\ h\ } \mathbb{A}^1 \xrightarrow{\ \eta\ } C \subset \mathbb{A}^2$$

where η is the normalization of C. It follows that $f, g \in K[h]$.

(c) It is clear that f, g are algebraically independent, i.e. $\operatorname{tdeg}_K K(f, g) = 2$. Equivalently, $K(x, y)/K(f, g)$ is a finite algebraic extension. Now assume that $\{h, f\} = \{h, g\} = 0$. Then the derivation D_h vanishes on $K[f, g]$, hence on $K[x, y]$. Thus $D_h = 0$ and so $h \in K$.

(d) Denote by $P_d := K[x, y]_d$ the homogeneous part of degree d. Let $L \subset P$ be the Lie subalgebra generated by $\{x, x^3, y^2\}$. We first use the equations

$$\{x, y\} = 1,\ \{x, y^2\} = 2y,\ \{x^3, y\} = 3x^2,\ \{x^2, y^2\} = 4xy,\ \{x^3, y^2\} = 6x^2y$$

to show that $K \oplus P_1 \oplus P_2 \subset L$ and that $x^2y \in L$. Now the claim follows by induction from the relations

$$\{x^n, x^2y\} = nx^{n+1} \quad\text{and}\quad \{x^ry^s, y^2\} = 2rx^{r-1}y^{s+1}.$$

□

Divergence. The next lemma should also be known. Recall that the *divergence* $\operatorname{Div} D$ of a vector field $D = p\partial_x + q\partial_y \in \operatorname{Vec}(\mathbb{A}^2)$ is defined by $\operatorname{Div} D := p_x + q_y \in K[x, y]$. Define

$$\operatorname{Vec}^0(\mathbb{A}^2) := \{D \in \operatorname{Vec}(\mathbb{A}^2) \mid \operatorname{Div} D = 0\} \subset \operatorname{Vec}^c(\mathbb{A}^2) := \{D \in \operatorname{Vec}(\mathbb{A}^2) \mid \operatorname{Div} D \in K\}.$$

The Lie algebra homomorphism $\mu\colon P \to \operatorname{Vec}(\mathbb{A}^2)$, $f \mapsto D_f$, has its image in $\operatorname{Vec}^0(\mathbb{A}^2)$, because $\operatorname{Div} D_f = 0$.

Lemma 2.2. *Let D be a non-trivial derivation of $K[x, y]$.*
(a) *The kernel $K[x, y]^D$ is either K or $K[f]$ for some $f \in K[x, y]$.*
(b) *If $\operatorname{Div} D = 0$, then $D = D_h$ for some $h \in K[x, y]$. In particular, $\mu(P) = \operatorname{Vec}^0(\mathbb{A}^2)$.*

Now assume that $D = D_f$ for some non-constant $f \in K[x, y]$ and that $D(g) = 1$ for some $g \in K[x, y]$.
(c) *Then $K[x, y]^D = K[f]$.*
(d) *If D is locally nilpotent, then $K[x, y] = K[f, g]$.*

Proof. (a) See [NN88] Theorem 2.8.

(b) Let $D = f\partial_x + g\partial_y$, then $\operatorname{Div} D = f_x + g_y = 0$ implies that there exists $h \in K[x, y]$ such that $f = h_y$, $g = -h_x$.

(c) It is obvious that $\ker(D) \supset K[f]$, hence, by (a), one has $\ker(D) = K[h] \supset K[f]$. Thus $f = F(h)$ for some $F \in K[t]$ and then $D_f(g) = D_{F(h)}(g) = F'(h)D_h(g) = 1$ which implies that F is linear.

(d) Let G be an affine algebraic group, X an affine variety and $\varphi\colon X \to G$ a G-equivariant retraction. Then one has $\mathcal{O}(X) = \varphi^*(\mathcal{O}(G)) \otimes \mathcal{O}(X)^G$. In our case we get $K[x, y] = \mathcal{O}(\mathbb{A}^2) = \mathcal{O}(G) \otimes \mathcal{O}(\mathbb{A}^2)^G = K[g] \otimes K[f]$. □

Automorphisms of the Poisson algebra. Denote by $\mathrm{Aut}_{LA}(P)$ the group of Lie algebra automorphisms of P. There is a canonical homomorphism

$$p\colon \mathrm{Aut}_{LA}(P) \to K^*, \quad \varphi \mapsto \varphi(1),$$

which has a section $s\colon K^* \to \mathrm{Aut}_{LA}(P)$ given by $s(t)|_{K[x,y]_n} := t^{1-n}\,\mathrm{id}_{K[x,y]_n}$ where $K[x,y]_n \subset K[x,y]$ denotes the subspace of homogeneous polynomials of degree n. Thus $\mathrm{Aut}_{LA}(P)$ is a semidirect product $\mathrm{Aut}_{LA}(P) = \mathrm{SAut}_{LA}(P) \rtimes K^*$ where

$$\mathrm{SAut}_{LA}(P) := \ker p = \{\alpha \in \mathrm{Aut}_{LA}(P) \mid \alpha(1) = 1\}.$$

Lemma 2.3. *Every automorphism $\alpha \in \mathrm{Aut}_{LA}(P)$ is determined by $\alpha(1)$, $\alpha(x)$ and $\alpha(y)$, and then $K[x,y] = K[\alpha(x), \alpha(y)]$.*

Proof. Replacing α by the composition $\alpha \circ s(\alpha(1)^{-1})$ we can assume that $\alpha(1) = 1$.

We will show that $\alpha(x^n) = \alpha(x)^n$ and $\alpha(y^n) = \alpha(y)^n$ for all $n \geq 0$. Then the first claim follows from Lemma 2.1(d).

By induction, we can assume that $\alpha(x^j) = \alpha(x)^j$ for $j < n$. We have $\{x^n, y\} = nx^{n-1}$ and so $\{\alpha(x^n), \alpha(y)\} = n\alpha(x^{n-1}) = n\alpha(x)^{n-1}$. On the other hand, we get $\{\alpha(x)^n, \alpha(y)\} = n\alpha(x)^{n-1}\{\alpha(x), \alpha(y)\} = n\alpha(x)^{n-1}$, hence the difference $h := \alpha(x^n) - \alpha(x)^n$ belongs to the kernel of the derivation $D_{\alpha(y)}\colon f \mapsto \{f, \alpha(y)\}$. Since $D_{\alpha(y)}$ is locally nilpotent, we get from Lemma 2.2(c)–(d) that $\ker D_{\alpha(y)} = K[\alpha(y)]$ and that $K[\alpha(x), \alpha(y)] = K[x,y]$. This already proves the second claim and shows that h is a polynomial in $\alpha(y)$.

Since $\{\alpha(x^n), \alpha(x)\} = \alpha(\{x^n, x\}) = 0$ and $\{\alpha(x)^n, \alpha(x)\} = n\alpha(x)^{n-1}\{\alpha(x), \alpha(x)\}$ we get $\{h, \alpha(x)\} = 0$ which implies that $h \in K$.

In the same way, using $\{x, xy\} = x$ and $\{y, xy\} = -y$, we find $\alpha(xy) - \alpha(x)\alpha(y) \in K$. Hence

$$n\alpha(x^n) = \{\alpha(x^n), \alpha(xy)\} = \{\alpha(x)^n, \alpha(x)\alpha(y)\} = n\alpha(x)^n,$$

and so $\alpha(x^n) = \alpha(x)^n$. By symmetry, we also get $\alpha(y^n) = \alpha(y)^n$. □

Automorphisms of affine 2-space. Denote by $\mathrm{Aut}(K[x,y])$ the group of K-algebra automorphisms of $K[x,y]$. We have a canonical identification $\mathrm{Aut}(\mathbb{A}^2) \xrightarrow{\sim} \mathrm{Aut}(K[x,y])^{op}$ given by $\varphi \mapsto \varphi^*$. For $\rho \in \mathrm{Aut}(K[x,y])$ we will use the notation $\rho = (f,g)$ in case $\rho(x) = f$ and $\rho(y) = g$, which implies that $K[x,y] = K[f,g]$. Note that the Jacobian determinant defines a homomorphism

$$j\colon \mathrm{Aut}(K[x,y]) \to K^*, \quad \rho \mapsto j(\rho) := j(\rho(x), \rho(y))$$

whose kernel is denoted by $\mathrm{SAut}(K[x,y])$.

We can consider $\mathrm{Aut}(K[x,y])$ and $\mathrm{Aut}_{LA}(P)$ as subgroups of the K-linear automorphisms $\mathrm{GL}(K[x,y])$.

Lemma 2.4. *As subgroups of $\mathrm{GL}(K[x,y])$ we have $\mathrm{SAut}_{LA}(P) = \mathrm{SAut}(K[x,y])$.*

Proof. (a) Let μ be an endomorphism of $K[x,y]$ and put $\mathrm{Jac}(\mu) := \mathrm{Jac}(\mu(x), \mu(y))$. For any $f, g \in K[x,y]$ we have $\mathrm{Jac}(\mu(f), \mu(g)) = \mu(\mathrm{Jac}(f,g))\,\mathrm{Jac}(\mu)$, because

$$\begin{aligned}\frac{\partial}{\partial x}\mu(f) &= \frac{\partial f}{\partial x}(\mu(x), \mu(y))\frac{\partial \mu(x)}{\partial x} + \frac{\partial f}{\partial y}(\mu(x), \mu(y))\frac{\partial \mu(y)}{\partial x} \\ &= \mu(\frac{\partial f}{\partial x})\frac{\partial \mu(x)}{\partial x} + \mu(\frac{\partial f}{\partial y})\frac{\partial \mu(y)}{\partial x}.\end{aligned}$$

It follows that $\{\mu(f), \mu(g)\} = \mu(\{f,g\})j(\mu)$. This shows that $\mathrm{SAut}(K[x,y]) \subset \mathrm{SAut}_{LA}(P)$.

(b) Now let $\alpha \in \mathrm{SAut}_{LA}(P)$. Then $j(\alpha(x), \alpha(y)) = \{\alpha(x), \alpha(y)\} = \alpha(1) = 1$ and, by Lemma 2.3, $K[\alpha(x), \alpha(y)] = K[x, y]$. Hence, we can define an automorphism $\rho \in \mathrm{SAut}(K[x, y])$ by $\rho(x) := \alpha(x)$ and $\rho(y) := \alpha(y)$. From (a) we see that $\rho \in \mathrm{SAut}_{LA}(P)$, and from Lemma 2.3 we get $\alpha = \rho$, hence $\alpha \in \mathrm{SAut}(K[x, y])$. □

Remark 2.5. The first part of the proof above shows the following. If $f, g \in P$ are such that $\{f, g\} = 1$, then the K-algebra homomorphism defined by $x \mapsto f$ and $y \mapsto g$ is an injective homomorphism of P as a Lie algebra. (Injectivity follows, because f, g are algebraically independent.)

Lie subalgebras of P. The subspace

$$P_{\leq 2} := K \oplus P_1 \oplus P_2 = K \oplus Kx \oplus Ky \oplus Kx^2 \oplus Kxy \oplus Ky^2 \subset P$$

is a Lie subalgebra. This can be deduced from the following Lie brackets which we note here for later use.

$$\{x^2, xy\} = 2x^2,\ \{x^2, y^2\} = 4xy,\ \{y^2, xy\} = -2y^2; \tag{1}$$

$$\{x^2, x\} = 0,\ \{xy, x\} = -x,\ \{y^2, x\} = -2y, \tag{2}$$

$$\{x^2, y\} = 2x,\ \{xy, y\} = y,\ \{y^2, y\} = 0; \tag{3}$$

$$\{x, y\} = 1. \tag{4}$$

Moreover, $P_2 = Kx^2 \oplus Kxy \oplus Ky^2$ is a Lie subalgebra of $P_{\leq 2}$ isomorphic to $\mathfrak{sl}_2$, and $P_1 = Kx \oplus Ky$ is the two-dimensional simple P_2-module.

From Remark 2.5 we get the following lemma.

Lemma 2.6. *Let $f, g \in K[x, y]$ such that $\{f, g\} = 1$. Then $\langle 1, f, g, f^2, fg, g^2\rangle \subset P$ is a Lie subalgebra isomorphic to $P_{\leq 2}$. An isomorphism is induced from the K-algebra homomorphism $P \to P$ defined by $x \mapsto f, y \mapsto g$.*

Definition 2.7. For $f, g \in K[x, y]$ such that $\{f, g\} \in K^*$ we put

$$P_{f,g} := \langle 1, f, g, f^2, fg, g^2\rangle \subset P.$$

We have just seen that this is a Lie algebra isomorphic to $P_{\leq 2}$. Clearly, $P_{f,g} = P_{f_1,g_1}$ if $\langle 1, f, g\rangle = \langle 1, f_1, g_1\rangle$. Denoting by $\mathfrak{rad}\, L$ the solvable radical of the Lie algebra L we get

$$\mathfrak{rad}\, P_{f,g} = \langle 1, f, g\rangle \quad \text{and} \quad P_{f,g}/\, \mathfrak{rad}\, P_{f,g} \simeq \mathfrak{sl}_2\,.$$

Proposition 2.8. *Let $Q \subset P$ be a Lie subalgebra isomorphic to $P_{\leq 2}$. Then $K \subset Q$, and $Q = P_{f,g}$ for every pair $f, g \in L$ such that $\langle 1, f, g\rangle = \mathfrak{rad}\, Q$. In particular, $\{f, g\} \in K^*$.*

Proof. We first show that $\mathfrak{cent}(Q) = K$. In fact, Q contains elements f, g such that $\{f, g\} \neq 0$. If $h \in \mathfrak{cent}(Q)$, then $h \in \mathfrak{cent}_P(f) \cap \mathfrak{cent}_P(g) = K$, by Lemma 2.1(c).

Now choose an isomorphism $\theta \colon P_{\leq 2} \xrightarrow{\sim} Q$. Then $\theta(K) = K$, and replacing θ by $\theta \circ s(t)$ with a suitable $t \in K^*$ we can assume that $\theta(1) = 1$. Setting $f := \theta(x), g := \theta(y)$ we get $\{f, g\} = 1$, and putting $f_0 := \theta(x^2), f_1 := \theta(xy), f_2 := \theta(y^2)$ we find

$$\{f_1, f\} = \theta\{xy, x\} = \theta(-x) = -f = \{fg, f\}.$$

Similarly, $\{f_1, g\} = \{fg, g\}$, hence $fg = f_1 + c \in Q$, by Lemma 2.1(c). Next we have

$$\{f_0, f\} = 0 \quad \text{and} \quad \{f_0, g\} = \theta(\{x^2, y\}) = \theta(2x) = 2f = \{f^2, g\}.$$

Hence $f^2 = f_0 + d$, and thus $f^2 \in Q$. A similar calculation shows that $g^2 \in Q$, so that we finally get $Q = P_{f,g}$. □

Characterization of $P_{\leq 2}$. The following lemma gives a characterization of the Lie algebras isomorphic to $P_{\leq 2}$.

Lemma 2.9. *Let Q be a Lie algebra containing a subalgebra Q_0 isomorphic to $\mathfrak{sl}_2$. Assume that*

(a) *$Q = Q_0 \oplus V_2 \oplus V_1$ as a Q_0-module where the V_i are simple of dimension i,*
(b) *V_1 is the center of Q, and*
(c) *$[V_2, V_2] = V_1$.*

Then Q is isomorphic to $P_{\leq 2}$.

Proof. Choosing an isomorphism of $P_2 = \langle x^2, xy, y^2 \rangle$ with Q_0 we find a basis (a_0, a_1, a_2) of Q_0 with relations

$$(1') \qquad [a_0, a_1] = 2a_0,\ [a_0, a_2] = 4a_1,\ [a_2, a_1] = -2a_2$$

(see (1) above). Since V_2 is a simple two-dimensional Q_0-module and $Kx \oplus Ky$ a simple two-dimensional P_2-module we can find a basis (b, c) of V_2 such that

$$(2') \qquad [a_0, b] = 0,\ [a_1, b] = -b,\ [a_2, b] = -2c,$$

$$(3') \qquad [a_0, c] = 2b,\ [a_1, c] = c,\ [a_2, c] = 0$$

(see (2) and (3) above). Finally, the last assumption (c) implies that

$$(4') \qquad d := [b, c] \neq 0,\ \text{hence } V_1 = Kd.$$

Comparing the relations (1)–(4) with (1′)–(4′) we see that the linear map $P_{\leq 2} \to Q$ given by $x^2 \mapsto a_0$, $xy \mapsto a_1$, $y^2 \mapsto a_2$, $x \mapsto b$, $y \mapsto c$, $1 \mapsto d$ is a Lie algebra isomorphism. □

3. Vector fields on affine 2-space

The action of $\mathrm{Aut}(\mathbb{A}^2)$ on vector fields. The group $\mathrm{Aut}(\mathbb{A}^2)$ acts on the vector fields $\mathrm{Vec}(\mathbb{A}^2)$. If $\varphi \in \mathrm{Aut}(\mathbb{A}^2)$ and if the vector fields $\mathrm{Vec}(\mathbb{A}^2)$ are regarded as sections $\xi\colon \mathbb{A}^2 \to T\mathbb{A}^2$ of the tangent bundle, then $\varphi^*(\xi) := (d\varphi)^{-1} \circ \xi \circ \varphi$. Writing $\xi = p\partial_x + q\partial_y$ and $\varphi = (f, g)$, we get

$$(*) \qquad \varphi^*(\xi) = \frac{1}{j(\varphi)} \left((g_y \varphi^*(p) - f_y \varphi^*(q))\, \partial_x + (-g_x \varphi^*(p) + f_x \varphi^*(q))\, \partial_y \right).$$

In particular,

$$\varphi^*(\partial_x) = \frac{1}{j(\varphi)}(g_y \partial_x - g_x \partial_y) \quad \text{and} \quad \varphi^*(\partial_y) = \frac{1}{j(\varphi)}(-f_y \partial_x + f_x \partial_y)$$

In fact, for every $u = (a, b) \in \mathbb{A}^2$ we have $d\varphi_u \circ \varphi^*(\xi)_u = \xi_{\varphi(u)}$. If $\varphi^*(\xi) = \tilde{p}\partial_x + \tilde{q}\partial_y$, this means that

$$\begin{bmatrix} f_x(u) & f_y(u) \\ g_x(u) & g_y(u) \end{bmatrix} \begin{bmatrix} \tilde{p}(u) \\ \tilde{q}(u) \end{bmatrix} = \begin{bmatrix} p(\varphi(u)) \\ q(\varphi(u)) \end{bmatrix}.$$

Hence

$$\begin{bmatrix} \tilde{p}(u) \\ \tilde{q}(u) \end{bmatrix} = \frac{1}{j(\varphi(u))} \begin{bmatrix} g_y(u) & -f_y(u) \\ -g_x(u) & f_x(u) \end{bmatrix} \begin{bmatrix} p(\varphi(u)) \\ q(\varphi(u)) \end{bmatrix}$$

and the claim follows.

Remark 3.1. If $\xi \in \mathrm{Vec}(\mathbb{A}^2)$ is considered as a derivation D of $K[x, y]$, and if $\alpha = \varphi^* \in \mathrm{Aut}(K[x, y])$, then the derivation corresponding to $\varphi^*(\xi)$ is given by $\alpha_* D = \alpha \circ D \circ \alpha^{-1}$.

Remark 3.2. If $\varphi\colon \mathbb{A}^2 \to \mathbb{A}^2$ is étale, i.e. $j(\varphi) \in K^*$, then the pull-back $\varphi^*(\xi)$ is well-defined for every vector field $\xi\colon \mathbb{A}^2 \to T\mathbb{A}^2$. It satisfies the equation $d\varphi \circ \varphi^*(\xi) = \xi \circ \varphi$ and it is given by the formula $(*)$. In terms of derivations, this corresponds to the well-known fact that for an étale extension $\alpha\colon A \hookrightarrow B$ every derivation D of A extends uniquely to a derivation of $\alpha_*(D)$ of B satisfying $\alpha_*(D) \circ \alpha = \alpha \circ D$.

It is not difficult to see that the map

$$\varphi^*\colon \operatorname{Vec}(\mathbb{A}^2) \to \operatorname{Vec}(\mathbb{A}^2),\ \xi \mapsto \varphi^*(\xi),$$

is an injective homomorphism of Lie algebras. In fact, if $\alpha = \varphi^* \in \operatorname{End}(K[x,y])$ and D the derivation of $K[x,y]$ that corresponds to ξ, then we find

$$\begin{aligned}\alpha_*([D_1,D_2]) \circ \alpha &= \alpha \circ [D_1,D_2] = \alpha \circ D_1 \circ D_2 - \alpha \circ D_2 \circ D_1 \\ &= \alpha_*(D_1) \circ \alpha \circ D_2 - \alpha_*(D_2) \circ \alpha \circ D_1 \\ &= \alpha_*(D_1) \circ \alpha_*(D_2) \circ \alpha - \alpha_*(D_2) \circ \alpha_*(D_1) \circ \alpha \\ &= [\alpha_*(D_1), \alpha_*(D_2)] \circ \alpha,\end{aligned}$$

hence the claim.

Recall that $\operatorname{Vec}^c(\mathbb{A}^2) \subset \operatorname{Vec}(\mathbb{A}^2)$ are the vector fields D with $\operatorname{Div} D \in K$. Clearly, the divergence $\operatorname{Div}\colon \operatorname{Vec}^c(\mathbb{A}^2) \to K$ is a character with kernel $\operatorname{Vec}^0(\mathbb{A}^2)$, and we have the decomposition

$$\operatorname{Vec}^c(\mathbb{A}^2) = \operatorname{Vec}^0(\mathbb{A}^2) \oplus KE \quad \text{where} \quad E := x\partial_x + y\partial_y \quad \text{is the } \textit{Euler field}.$$

Lemma 3.3. *If $\varphi\colon \mathbb{A}^2 \to \mathbb{A}^2$ is étale, then $\varphi^*(D_h) = j(\varphi)^{-1} D_{\varphi^*(h)}$. Moreover, $\operatorname{Div}(\varphi^*(E)) = 2$, and so $\varphi^*(\operatorname{Vec}^0(\mathbb{A}^2)) \subset \operatorname{Vec}^0(\mathbb{A}^2)$ and $\varphi^*(\operatorname{Vec}^c(\mathbb{A}^2)) \subset \operatorname{Vec}^c(\mathbb{A}^2)$. In particular, the homomorphism $\mu\colon P \to \operatorname{Vec}(\mathbb{A}^2)$ is equivariant with respect to the group $\operatorname{SAut}(K[x,y]) = \operatorname{SAut}_{LA}(P)$.*

Proof. Put $\alpha := \varphi^* \in \operatorname{End}(K[x,y])$. We have $\alpha(D_h) \circ \alpha = \alpha \circ D_h$, hence

$$\begin{aligned}\alpha(D_h)(\alpha(f)) = \alpha(D_h(f)) = \alpha(j(h,f)) = j(\alpha)^{-1} j(\alpha(h), \alpha(f)) = \\ = j(\alpha)^{-1} D_{\alpha(h)}(\alpha(f)).\end{aligned}$$

From formula $(*)$ we get $\alpha(E) = \frac{1}{j(\alpha)}\left((g_y f - f_y g)\partial_x + (-g_x f + f_x g)\partial_y\right)$ which implies that $\operatorname{Div} \alpha(E) = 2$. □

Remark 3.4. Let $\varphi\colon \mathbb{A}^2 \to \mathbb{A}^2$ be étale. If $\varphi^*\colon \operatorname{Vec}^0(\mathbb{A}^2) \to \operatorname{Vec}^0(\mathbb{A}^2)$ is an isomorphism, then so is φ. In fact, $\varphi^*(D_{c\cdot h}) = D_{\varphi^*(h)}$ for $c := j(\varphi) \in K^*$, showing that every $f \in K[x,y]$ is of the form $\varphi^*(h)$ up to a constant. It follows that $\varphi^*\colon K[x,y] \to K[x,y]$ is surjective, hence an isomorphism.

Remark 3.5. The lemma above implies that we have canononical homomorphisms

$$\begin{gathered}\operatorname{Aut}(K[x,y]) \to \operatorname{Aut}_{LA}(\operatorname{Vec}(\mathbb{A}^2)), \\ \operatorname{Aut}(K[x,y]) \to \operatorname{Aut}_{LA}(\operatorname{Vec}^c(\mathbb{A}^2)), \\ \operatorname{Aut}(K[x,y]) \to \operatorname{Aut}_{LA}(\operatorname{Vec}^0(\mathbb{A}^2)).\end{gathered}$$

We will see in Theorem 4.5 that these are all isomorphisms.

Lie subalgebras of $\mathrm{Vec}(\mathbb{A}^2)$**.** Let $\mathrm{Aff}(\mathbb{A}^2)$ denote the group of *affine transformations* of $\mathbb{A}^2$, $x \mapsto Ax + b$, where $A \in \mathrm{GL}_2(K)$ and $b \in K^2$. The determinant defines a character $\det\colon \mathrm{Aff}(\mathbb{A}^2) \to K^*$ whose kernel will be denoted by $\mathrm{SAff}(\mathbb{A}^2)$. For the corresponding Lie algebras we write $\mathfrak{saff}_2 := \mathrm{Lie}\,\mathrm{SAff}(\mathbb{A}^2) \subset \mathfrak{aff}_2 := \mathrm{Lie}\,\mathrm{Aff}(\mathbb{A}^2)$. There is a canonical embedding $\mathfrak{aff}_2 \subset \mathrm{Vec}(\mathbb{A}^2)$ which identifies $\mathfrak{aff}_2$ with the Lie subalgebra

$$\langle \partial_x, \partial_y, x\partial_x + y\partial_y, x\partial_x - y\partial_y, x\partial_y, y\partial_x \rangle \subset \mathrm{Vec}^c(\mathbb{A}^2),$$

and $\mathfrak{saff}_2$ with

$$\mu(P_{x,y}) = \langle \partial_x, \partial_y, x\partial_x - y\partial_y, x\partial_y, y\partial_x \rangle \subset \mathrm{Vec}^0(\mathbb{A}^2).$$

Note that the Euler field $E = x\partial_x + y\partial_y \in \mathfrak{aff}_2$ is determined by the condition that E acts trivially on $\mathfrak{sl}_2$ and that $[E, D] = -D$ for $D \in \mathfrak{rad}(\mathfrak{saff}_2) = K\partial_x \oplus K\partial_y$. We also remark that the centralizer of $\mathfrak{saff}_2$ in $\mathrm{Vec}(\mathbb{A}^2)$ is trivial:

$$\mathfrak{cent}_{\mathrm{Vec}(\mathbb{A}^2)}(\mathfrak{saff}_2) = (0).$$

In fact, $\mathfrak{cent}_{\mathrm{Vec}(\mathbb{A}^2)}(\{\partial_x, \partial_y)\} = K\partial_x \oplus K\partial_y$, and $(K\partial_x \oplus K\partial_y)^{\mathfrak{sl}_2} = (0)$.

Let $\varphi = (f, g)\colon \mathbb{A}^2 \to \mathbb{A}^2$ be étale, and assume, for simplicity, that $j(f, g) = 1$. From formula $(*)$ we get

$$\begin{gathered}
\varphi^*(\partial_x) = g_y\partial_x - g_x\partial_y = -D_g, \quad \varphi^*(\partial_y) = -f_y\partial_x + f_x\partial_y = D_f,\\
\varphi^*(x\partial_y) = fD_f = \tfrac{1}{2}D_{f^2}, \quad \varphi^*(y\partial_x) = -gD_g = -\tfrac{1}{2}D_{g^2},\\
\varphi^*(x\partial_x) = -fD_g, \quad \varphi^*(y\partial_y) = gD_f, \quad \varphi^*(x\partial_x - y\partial_y) = -D_{fg}.
\end{gathered}$$

This shows that for an étale map $\varphi = (f, g)$ we obtain

$$\begin{gathered}
\varphi^*(\mathfrak{aff}_2) = \langle D_f, D_g, D_{f^2}, D_{g^2}, fD_g, gD_f \rangle,\\
\varphi^*(\mathfrak{saff}_2) = \langle D_f, D_g, D_{f^2}, D_{g^2}, D_{fg} \rangle = \mu(P_{f,g})
\end{gathered}$$

Proposition 3.6. *Let* $L \subset \mathrm{Vec}^c(\mathbb{A}^2)$ *be a Lie subalgebra isomorphic to* $\mathfrak{saff}_2$*. Then there is an étale map* φ *such that* $L = \varphi^*(\mathfrak{saff}_2)$*. More precisely, if* (D_f, D_g) *is a basis of* $\mathfrak{rad}(L)$*, then* $L = \langle D_f, D_g, D_{f^2}, D_{g^2}, D_{fg} \rangle$*, and one can take* $\varphi = (f, g)$*.*

Proof. We first remark that $L \subset \mathrm{Vec}^0(\mathbb{A}^2)$, because $\mathfrak{saff}_2$ has no non-trivial character. By Proposition 2.8 it suffices to show that $Q := \mu^{-1}(L) \subset P$ is isomorphic to $P_{\leq 2}$. We fix a decomposition $L = L_0 \oplus \mathfrak{rad}(L)$ where $L_0 \simeq \mathfrak{sl}_2$. It is clear that the Lie subalgebra $\tilde{Q} := \mu^{-1}(L_0) \subset P$ contains a copy of $\mathfrak{sl}_2$, i.e. $\tilde{Q} = Q_0 \oplus K$ where $Q_0 \simeq \mathfrak{sl}_2$. Hence, as a Q_0-module, we get $Q = Q_0 \oplus V_2 \oplus K$ where V_2 is a two-dimensional irreducible Q_0-module which is isomorphically mapped onto $\mathfrak{rad}(L)$ under μ. Since $\{\mathfrak{rad}(L), \mathfrak{rad}(L)\} = (0)$ we have $\{V_2, V_2\} \subset K$. Now the claim follows from Lemma 2.9 if we show that $\{V_2, V_2\} \neq (0)$.

Assume that $\{V_2, V_2\} = (0)$. Choose a $\mathfrak{sl}_2$-triple (e_0, h_0, f_0) in Q_0 and a basis (f, g) of V_2 such that $\{e_0, f\} = g$ and $\{e_0, g\} = 0$. Since $\{f, g\} = 0$ we get from Lemma 2.1(b) that $f, g \in K[h]$ for some $h \in K[x, y]$, i.e. $f = p(h)$ and $g = q(h)$ for some polynomials $p, q \in K[t]$. But then $0 = \{e_0, g\} = \{e_0, q(h)\} = q'(h)\{e_0, h\}$ and so $\{e_0, h\} = 0$. This implies that $g = \{e_0, f\} = \{e_0, p(h)\} = p'(h)\{e_0, h\} = 0$, a contradiction. $\square$

Remark 3.7. The above description of the Lie subalgebras L isomorphic to $\mathfrak{saff}_2$ also gives a Levi decomposition of L. In fact, (D_f, D_g) is a basis of $\mathfrak{rad}(L)$ and $L_0 := \langle D_{f^2}, D_{g^2}, D_{fg} \rangle$ is a subalgebra isomorphic to $\mathfrak{sl}_2$. The following corollary shows that every Levi decomposition is obtained in this way.

Corollary 3.8. *Let $L \subset \mathrm{Vec}^c(\mathbb{A}^2)$ be a Lie subalgebra isomorphic to $\mathfrak{saff}_2$, and let $L = \mathfrak{rad}(L) \oplus L_0$ be a Levi decomposition. Then there exist $f, g \in K[x,y]$ such that $\mathfrak{rad}(L) = \langle D_f, D_g \rangle$ and $L_0 = \langle D_{f^2}, D_{fg}, D_{g^2} \rangle$. Moreover, if $L' \subset \mathrm{Vec}^c(\mathbb{A}^2)$ is another Lie subalgebra isomorphic to $\mathfrak{saff}_2$ and if $L' \supset L_0$, then $L' = L$.*

Proof. We can assume that $L = \mathfrak{saff}_2 = \langle D_x, D_y, D_{x^2}, D_{y^2}, D_{xy} \rangle$. Then every Lie subalgebra $L_0 \subset L$ isomorphic to $\mathfrak{sl}_2$ is the image of $\mathfrak{sl}_2 = \langle D_{x^2}, D_{y^2}, D_{xy} \rangle$ under conjugation with an element α of the solvable radical R of SAff_2. As a subgroup of $\mathrm{Aut}(K[x,y])$ the elements of R are the translations $\alpha = (x+a, y+b)$, and we get $\mathfrak{rad}(L) = \langle D_{x+a}, D_{y+b} \rangle$ and $\alpha(\mathfrak{sl}_2) = \langle D_{(x+a)^2}, D_{(y+b)^2}, D_{(x+a)(y+b)} \rangle$ as claimed.

For the last statement, we can assume that $L' = \langle D_f, D_g, D_{f^2}, D_{g^2}, D_{fg} \rangle$ such that $\langle D_{f^2}, D_{g^2}, D_{fg} \rangle = \mathfrak{sl}_2$. This implies that $\langle f^2, g^2, fg, 1 \rangle = \langle x^2, y^2, xy, 1 \rangle$, and the claim follows. ☐

Proposition 3.9. *Let $M \subset \mathrm{Vec}^c(\mathbb{A}^2)$ be a Lie subalgebra isomorphic to $\mathfrak{aff}_2$. Then there is an étale map φ such that $M = \varphi^*(\mathfrak{aff}_2)$. More precisely, if (D_f, D_g) is a basis of $\mathfrak{rad}([M,M])$, then $M = \langle D_f, D_g, fD_f, gD_g, gD_f, fD_g \rangle$, and one can take $\varphi = (f,g)$.*

Proof. The subalgebra $M' := [M,M]$ is isomorphic to $\mathfrak{saff}$, hence, by Proposition 3.6, $M' = \varphi^*(\mathfrak{saff}_2)$ for an étale map $\varphi = (f,g)$ where we can assume that $j(\alpha) = 1$. We want to show that $\varphi^*(\mathfrak{aff}_2) = M$. Consider the decomposition $M = J \oplus M_0 \oplus KD$ where $J = \mathfrak{rad}(M')$, M_0 is isomorphic to $\mathfrak{sl}_2$, and D is the Euler-element acting trivially on M_0. We have $\varphi^*(\mathfrak{aff}_2) = M' \oplus KE$ where E is the image of the Euler element of $\mathfrak{aff}_2$. Since $\mathrm{Vec}^c(\mathbb{A}^2) = \mathrm{Vec}^0(\mathbb{A}^2) \oplus KD'$ for any $D' \in \mathrm{Vec}^c(\mathbb{A}^2)$ with $\mathrm{Div}\, D' \neq 0$ we can write $D = aE + F$ with some $a \in K$ and $F \in \mathrm{Vec}^0(\mathbb{A}^2)$, i.e. $F = D_h$ for some $h \in K[x,y]$.

By construction, $F = D - aE$ commutes with M_0. Since $M_0 = \langle D_{f^2}, D_{g^2}, D_{fg} \rangle$ we get $\{h, f^2\} = c$ where $c \in K$. Thus $c = \{h, f^2\} = 2f\{h,f\}$ which implies that $\{h,f\} = 0$. Similarly, we find $\{h,g\} = 0$, hence h is in the center of $\mu^{-1}(M') = P_{f,g} \subset P$. Thus, by Lemma 2.1(c), $h \in K$ and so $D_h = 0$ which implies $D = aE$. ☐

4. Vector fields and the Jacobian Conjecture

The Jacobian Conjecture. Recall that the *Jacobian Conjecture* in dimension n says that an étale morphism $\varphi\colon \mathbb{A}^n \to \mathbb{A}^n$ is an isomorphism.

Theorem 4.1. *The following statements are equivalent.*

(i) *The Jacobian Conjecture holds in dimension 2.*
(ii) *All Lie subalgebras of P isomorphic to $P_{\leq 2}$ are equivalent under $\mathrm{Aut}_{LA}(P)$.*
(iii) *All Lie subalgebras of $\mathrm{Vec}^c(\mathbb{A}^2)$ isomorphic to $\mathfrak{saff}_2$ are conjugate under $\mathrm{Aut}(\mathbb{A}^2)$.*
(iv) *All Lie subalgebras of $\mathrm{Vec}^c(\mathbb{A}^2)$ isomorphic to $\mathfrak{aff}_2$ are conjugate under $\mathrm{Aut}(\mathbb{A}^2)$.*

For the proof we need to compare the automorphisms of P with those of the image $\mu(P) = \mathrm{Vec}^0(\mathbb{A}^2) \simeq P/K$. Since K is the center P, we have a canonical homomorphism $F\colon \mathrm{Aut}_{LA}(P) \to \mathrm{Aut}_{LA}(P/K)$, $\varphi \mapsto \bar{\varphi}$.

Lemma 4.2. *The map $F\colon \mathrm{Aut}_{LA}(P) \to \mathrm{Aut}_{LA}(P/K)$ is an isomorphism.*

Proof. If $\varphi \in \ker F$, then $\varphi(x) = x+a, \varphi(y) = y+b$ where $a, b \in K$. By Lemma 2.4, the K-algebra automorphism α of $K[x,y]$ defined by $x \mapsto x+a$, $y \mapsto y+b$ is a Lie algebra automorphism of P, and $\varphi = \alpha$ by Lemma 2.3. But then $\varphi(x^2) = (x+a)^2 = x^2 + 2ax + a^2$, and so $\bar{\varphi}(\overline{x^2}) = \overline{x^2} + 2a\bar{x}$. Therefore, $a = 0$, and similarly we get $b = 0$, hence $\varphi = \mathrm{id}_P$.

Put $\bar{P} := P/K$ and let $\rho\colon \bar{P} \xrightarrow{\sim} \bar{P}$ be a Lie algebra automorphism. Then $\bar{L} := \rho(\bar{P}_{\leq 2}) \subset \bar{P}$ is a Lie subalgebra isomorphic to $\mathfrak{saff}_2$ and thus $L := p^{-1}(\bar{L})$ is a Lie subalgebra of P isomorphic to $P_{\leq 2}$, by Proposition 2.8. Choose $f, g \in L$ such that $\bar{f} = \rho(\bar{x})$ and $\bar{g} = \rho(\bar{y})$. Then $\langle 1, f, g\rangle = \mathfrak{rad}(L)$, and so $L = P_{f,g}$, by Proposition 2.8. It follows that the map $\mu\colon P \to P$ defined by $x \mapsto f, y \mapsto g$ is an injective endomorphism of P (Remark 2.5), and that $\bar{\mu} = \rho$. Since ρ is an isomorphism the same holds for μ. □

Proof of Theorem 4.1. (i)⇒(ii): If $L \subset P$ is isomorphic to $P_{\leq 2}$, then $L = P_{f,g}$ for some $f, g \in K[x,y]$ such that $\{f, g\} = 1$ (Proposition 2.8). By (i) we get $K[x,y] = K[f,g]$, and so the endomorphism $x \mapsto f, y \mapsto g$ of $K[x,y]$ is an isomorphism of P, mapping $P_{\leq 2}$ to L.

(ii)⇒(iii): If $\bar{L} \subset \mathrm{Vec}^c(\mathbb{A}^2)$ is a Lie subalgebra isomorphic to $\mathfrak{saff}_2$, then $\bar{L} = \mu(P_{f,g})$ for some $f, g \in K[x,y]$, by Proposition 3.6. By (ii), $P_{f,g} = \alpha_*(P_{\leq 2})$ for some $\alpha \in \mathrm{SAut}_{LA}(P) = \mathrm{SAut}(K[x,y])$. Hence $\bar{L} = \mu(\alpha_*(\bar{P}_{\leq 2})) = \bar{\alpha}(\mathfrak{saff}_2)$, by Lemma 3.3.

(iii)⇒(iv): Let $M \subset \mathrm{Vec}^c(\mathbb{A}^2)$ be a Lie subalgebra isomorphic to $\mathfrak{aff}_2$, and set $M' := [M, M] \simeq \mathfrak{saff}_2$. By (iii) there is an automorphism $\varphi \in \mathrm{Aut}(\mathbb{A}^2)$ such that $M' = \varphi^*(\mathfrak{saff}_2)$. It follows that $\varphi^*(\mathfrak{aff}_2) = M$ since M is determined by $\mathfrak{rad}(M')$ as a Lie subalgebra, by Proposition 3.9.

(iv)⇒(i): Let $\varphi := (f,g)\colon \mathbb{A}^2 \to \mathbb{A}^2$ be an étale morphism. Then $M := \varphi^*(\mathfrak{aff}_2) \subset \mathrm{Vec}^c(\mathbb{A}^2)$ is a Lie subalgebra isomorphic to $\mathfrak{aff}_2$ (see Lemma 3.3). By assumption (iv), there is an automorphism $\psi \in \mathrm{Aut}(\mathbb{A}^2)$ such that $\psi^*(\mathfrak{aff}_2) = M$. It follows that $\psi^{-1} \circ \varphi$ is an étale morphism which induces an automorphism of $\mathfrak{aff}_2$, hence of $\mathfrak{saff}_2$, and thus of $\mathfrak{rad}(\mathfrak{saff}_2) = K\partial_x \oplus K\partial_y$. This implies that $\psi^{-1} \circ \varphi$ is an automorphism, and the claim follows. □

Remark 4.3. It is not true that the Lie subalgebras of P or of $\mathrm{Vec}^c(\mathbb{A}^2)$ isomorphic to $\mathfrak{sl}_2$ are equivalent, respectively conjugate. This can be seen from the example $S = Kx^2y \oplus Kxy \oplus Ky \subset P$ which is isomorphic to $\mathfrak{sl}_2$, but not equivalent to $Kx^2 \oplus Kxy \oplus Ky^2$ under $\mathrm{Aut}_{LA}(P)$. In fact, the element x^2y does not act locally finitely on P.

Algebraic Lie algebras. If an algebraic group G acts on an affine variety X we get a canonical anti-homomorphism of Lie algebras $\Phi\colon \mathrm{Lie}\, G \to \mathrm{Vec}(X)$ defined in the usual way:

$$\mathrm{Lie}\, G \ni A \mapsto \xi_A \text{ with } (\xi_A)_x := d\varphi_x(A) \text{ for } x \in X,$$

where $\varphi_x\colon G \to X$ is the orbit map $g \mapsto gx$. A Lie algebra $L \subset \mathrm{Vec}(X)$ is called *algebraic* if L is contained in $\Phi(\mathrm{Lie}\, G)$ for some action of an algebraic group G on X. It is shown in [CD03] that L is algebraic if and only if L acts locally finitely on $\mathrm{Vec}(X)$. With this result we get the following consequence of our Theorem 1.

Corollary 4.4. *The following statements are equivalent.*

(i) *The Jacobian Conjecture holds in dimension 2.*
(ii) *All Lie subalgebras of* $\mathrm{Vec}^c(\mathbb{A}^2)$ *isomorphic to* $\mathfrak{saff}_2$ *are algebraic.*

(iii) *All Lie subalgebras of* $\mathrm{Vec}^c(\mathbb{A}^2)$ *isomorphic to* $\mathfrak{aff}_2$ *are algebraic.*

Proof. It is clear that the equivalent statements (i), (ii) or (iii) of Theorem 1 imply (ii) and (iii) from the corollary. It follows from the Propositions 3.6 and 3.9 that every Lie subalgebra L isomorphic to $\mathfrak{saff}_2$ is contained in a Lie subalgebra Q isomorphic to $\mathfrak{aff}_2$, hence (iii) implies (ii). It remains to prove that (ii) implies (i).

We will show that (ii) implies that L is equivalent to $\mathfrak{saff}_2$. Then the claim follows from Theorem 1. By (ii), there is a connected algebraic group G acting faithfully on $\mathbb{A}^2$ such that $\Phi(\mathrm{Lie}\, G)$ contains L. Therefore, $\mathrm{Lie}\, G$ contains a subalgebra $\mathfrak{s}$ isomorphic to $\mathfrak{sl}_2$, and so G contains a closed subgroup S such that $\mathrm{Lie}\, S = \mathfrak{s}$. Since every action of SL_2 on $\mathbb{A}^2$ is linearizable (see [KP85]), there is an automorphism φ such that $\varphi^*(\mathfrak{s}) = \mathfrak{sl}_2 = \langle x\partial_y, y\partial_x, x\partial_x - y\partial_y\rangle$. But this implies, by Corollary 3.8, that $\varphi^*(L) = \mathfrak{saff}_2$. $\square$

Automorphisms of vector fields. We have seen in Lemma 2.4 that $\mathrm{SAut}_{LA}(P) = \mathrm{SAut}(K[x,y])$. In this last section we describe the automorphism groups of the Lie algebras $\mathrm{Vec}(\mathbb{A}^2)$, $\mathrm{Vec}^c(\mathbb{A}^2)$ and $\mathrm{Vec}^0(\mathbb{A}^2)$.

Theorem 4.5. *There are canonical isomorphisms*

$$\mathrm{Aut}(\mathbb{A}^2) \xrightarrow{\sim} \mathrm{Aut}_{LA}(\mathrm{Vec}(\mathbb{A}^2)) \xrightarrow{\sim} \mathrm{Aut}_{LA}(\mathrm{Vec}^c(\mathbb{A}^2)) \xrightarrow{\sim} \mathrm{Aut}_{LA}(\mathrm{Vec}^0(\mathbb{A}^2)).$$

For the proof we need the following two results. The first one is certainly well-known. Recall that $\mathfrak{saff}_2 = [\mathfrak{aff}_2, \mathfrak{aff}_2] \subset \mathfrak{aff}_2$ is invariant under all automorphisms of the Lie algebra $\mathfrak{aff}_2$.

Lemma 4.6. *The canonical homomorphisms*

$$\mathrm{Aff}_2 \xrightarrow[\simeq]{\mathrm{Ad}} \mathrm{Aut}_{LA}(\mathfrak{aff}_2) \xrightarrow[\simeq]{\mathrm{res}} \mathrm{Aut}_{LA}(\mathfrak{saff}_2)$$

are isomorphisms.

Proof. We write the elements of Aff_2 in the form (v,g) with $v \in T = (K^+)^2$ and $g \in \mathrm{GL}_2$ where $(v,g)x = gx+v$ for $x \in \mathbb{A}^2$. It follows that $(v,g)(w,h) = (v+gw, gh)$. Similarly, $(a,A) \in \mathfrak{aff}_2$ means that $a \in \mathbf{t} = (K)^2$ and $A \in \mathfrak{gl}_2$, and $(a,A)x = Ax+a$. For the adjoint representation of $g \in \mathrm{GL}_2$ and of $v \in T$ on $\mathfrak{aff}_2$ we get

$$\mathrm{Ad}(g)(a,A) = (ga, gAg^{-1}) \quad \text{and} \quad \mathrm{Ad}(v)(a,A) = (a - Av, A),$$

and thus, for $(b,B) \in \mathfrak{aff}_2$,

$$(**) \qquad \mathrm{ad}(B)(a,A) = (Ba, [B,A]) \quad \text{and} \quad \mathrm{ad}(b)(a,A) = (a - Ab, A).$$

Now let θ be an automorphism of the Lie algebra $\mathfrak{saff}_2$. Then $\theta(\mathbf{t}) = \mathbf{t}$, because $\mathbf{t}$ is the solvable radical of $\mathfrak{saff}_2$. Since $g := \theta|_{\mathbf{t}} \in \mathrm{GL}_2$, composing θ with $\mathrm{Ad}(g^{-1})$, we can assume that θ is the identity on $\mathbf{t}$. This implies that $\theta(a,A) = (a + \ell(A), \bar\theta(A))$ where $\ell\colon \mathfrak{sl}_2 \to \mathbf{t}$ is a linear map and $\bar\theta\colon \mathfrak{sl}_2 \to \mathfrak{sl}_2$ is a Lie algebra automorphism.

From $(**)$ we get $\mathrm{ad}(b,B)(a,0) = \mathrm{ad}(B)(a,0) = (Ba,0)$ for all $a \in \mathbf{t}$, hence

$$(Ba,0) = \theta(Ba,0) = \theta(\mathrm{ad}(B)(a,0)) = \\ = \mathrm{ad}(\theta(B))(a,0) = \mathrm{ad}(\bar\theta(B))(a,0) = (\bar\theta(B)a, 0).$$

Thus $\bar\theta(B) = B$, i.e. $\theta(a,A) = (a + \ell(A), A)$. For $c := \ell(E)$ we obtain

$$\theta(a, \lambda E) = (a + \lambda c, \lambda E) = \mathrm{Ad}(-c)(a, \lambda E).$$

Thus we can assume that θ is the identity on $KE \subset \mathfrak{aff}_n$. Since M_n is the centralizer of KE in $\mathfrak{aff}_n$ this implies that $\theta(\mathrm{M}_n) = \mathrm{M}_n$, hence $\theta(0, A) = (0, \theta(A)) = (0, \bar{\theta}(A)) = (0, A)$. As a consequence, $\theta = \mathrm{id}$, and the claim follows. $\square$

Lemma 4.7. *If θ is an endomorphism of the Lie algebra $\mathrm{Vec}^0(\mathbb{A}^2)$ which is the identity on $\mathfrak{saff}_2$, then θ is the identity.*

Proof. It follows from Lemma 2.1(d) and Lemma 2.2(b) that $\mathrm{Vec}^0(\mathbb{A}^2)$ is generated by the vector fields ∂_y, $x^2\partial_y$, and $y\partial_x$. So it suffices to show that $\theta(x^2dy) = x^2dy$.

Put $D := \theta(x^2dy)$. Since $[\partial_y, D] = \theta([\partial_y, x^2\partial_y]) = 0$ we see that $D = h(x)\partial_x + f(x)\partial_y$. But $0 = \mathrm{Div}\, D = h_x$, and so $D = a\partial_x + f(x)\partial_y$.

Now $[\partial_x, D] = \theta([\partial_x, a\partial_x + x^2\partial_y]) = \theta(2x\partial_y) = 2x\partial_y = [\partial_x, x^2\partial_y]$. Hence $D = a\partial_x + x^2\partial_y + b\partial_y$. Finally, $[x\partial_y, D] = -a\partial_y = \theta([x\partial_y, x^2\partial_y]) = 0$, hence $a = 0$, and similarly, $[y\partial_x, D] = 2x\partial_y - b\partial_x = \theta([y\partial_x, x^2\partial_y]) = \theta(2x\partial_y) = 2x\partial_y$, hence $b = 0$. $\square$

Proof of Theorem 4.5. (a) The fact that $\mathrm{Aut}(\mathbb{A}^2) \to \mathrm{Aut}_{LA}(\mathrm{Vec}(\mathbb{A}^2))$ is an isomorphism goes back to Kulikov (see proof of theorem 4, [Kul92]). For another proof see [Bav13].

(b) It follows from (a) that we have a canonical homomorphism, by restriction,

$$\mathrm{Aut}_{LA}(\mathrm{Vec}(\mathbb{A}^2)) \to \mathrm{Aut}_{LA}(\mathrm{Vec}^c(\mathbb{A}^2)),$$

and since $\mathrm{Vec}^0(\mathbb{A}^2) \subset \mathrm{Vec}^c(\mathbb{A}^2)$ is an ideal of finite codimension and is simple as a Lie algebra we also get a homomorphism

$$\mathrm{Aut}_{LA}(\mathrm{Vec}^c(\mathbb{A}^2)) \to \mathrm{Aut}_{LA}(\mathrm{Vec}^0(\mathbb{A}^2))$$

which is easily seen to be injective. Thus it remains to show that the canonical homomorphism $\omega\colon \mathrm{Aut}(\mathbb{A}^2) \to \mathrm{Aut}_{LA}(\mathrm{Vec}^0(\mathbb{A}^2))$ is an isomorphism.

(c) It is clear that ω is injective. Let θ be an automorphism of $\mathrm{Vec}^0(\mathbb{A}^2)$. It follows from Proposition 3.6 that there is an étale map φ such that $\varphi^*(\mathfrak{saff}_2) = \theta(\mathfrak{saff}_2)$. Hence the homomorphism $\theta^{-1} \circ \varphi^*$ maps $\mathfrak{saff}_2$ isomorphically onto itself. This implies, by Lemma 4.6, that $(\theta^{-1} \circ \varphi^*)|_{\mathfrak{saff}_2} = \mathrm{Ad}(\psi)$ that for a suitable $\psi \in \mathrm{Aff}_2$. By definition, $\psi^*|_{\mathfrak{saff}_2} = \mathrm{Ad}(\psi)^{-1}$, and so the composition $\theta^{-1} \circ \varphi^* \circ \psi^*$ is the identity on $\mathfrak{saff}_2$, hence the identity on $\mathrm{Vec}^0(\mathbb{A}^2)$, by Lemma 4.7. Therefore, by Remark 3.4, φ is an isomorphism, and so $\theta = \varphi^* \circ \psi^*$ belongs to the image of $\omega\colon \mathrm{Aut}(\mathbb{A}^2) \to \mathrm{Aut}_{LA}(\mathrm{Vec}^0(\mathbb{A}^2))$. $\square$

Remark 4.8. In [KReg14] our Theorem 4.5 is generalized to any dimension, using a completely different approach.

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Mathematisches Institut, Universität Basel, Spiegelgasse 1, CH-4051 Basel
E-mail address: `Andriy.Regeta@unibas.ch`

CHARACTERIZATION OF n-DIMENSIONAL NORMAL AFFINE SL_n-VARIETIES

ANDRIY REGETA

Abstract. We show that any normal irreducible affine n-dimensional SL_n-variety X is determined by its automorphism group in the category of normal irreducible affine varieties: if Y is an irreducible affine normal algebraic variety such that $\mathrm{Aut}(X) \cong \mathrm{Aut}(Y)$ as ind-groups, then $Y \cong X$ as varieties. If we drop the condition of normality on Y, then X is not uniquely determined and we classify all such varieties. In case $n \geq 3$, all the above results hold true if we replace $\mathrm{Aut}(X)$ by $U(X)$, where $U(X)$ is the subgroup of $\mathrm{Aut}(X)$ generated by all one-dimensional unipotent subgroups. In dimension 2 we have some very interesting exceptions.

1. Introduction and Main Results

Our base field is the field of complex numbers $\mathbb{C}$. For an affine variety X the automorphism group $\mathrm{Aut}(X)$ has the structure of an ind-group. We will shortly recall the basic definitions and results in Section 2. The classical example is $\mathrm{Aut}(\mathbb{A}^n)$, $n > 1$, the group of automorphisms of the affine n-space $\mathbb{A}^n$. Recently, Hanspeter Kraft proved the following result which shows that the affine n-space is determined by its automorphism group (see [Kr15]).

Theorem 0. *Let Y be a connected affine variety. If $\mathrm{Aut}(Y) \cong \mathrm{Aut}(\mathbb{A}^n)$ as ind-groups, then $Y \cong \mathbb{A}^n$ as varieties.*

In this paper we prove a similar result for some other varieties which we are going to define now. Let $d > 1$. Consider the action of $\mu_d = \{\xi \in \mathbb{C}^* | \xi^d = 1\}$ on $\mathbb{A}^n$ by scalar multiplication and denote by $\pi : \mathbb{A}^n \to A_{d,n} := \mathbb{A}^n/\mu_d$ the quotient. This means that $A_{d,n}$ is an affine variety with coordinate ring $\mathcal{O}(A_{d,n}) = \mathbb{C}[x_1, ..., x_n]^{\mu_d}$, the algebra of invariants (see [Mu74]). Note that $A_{d,n}$ is indeed an orbit space, because μ_d is finite. For $d > 1$, $A_{d,n}$ has an isolated singularity in $\pi(0)$ and π induces an étale covering $\mathbb{A}^n \setminus \{0\} \to A_{d,n} \setminus \{p(0)\}$ with Galois group μ_d. Later on we consider only the case $d > 1$.

Theorem 1. *Let X be a normal affine variety such that $\mathrm{Aut}(X) \cong \mathrm{Aut}(A_{d,n})$ as ind-group, then we have an isomorphism $X \cong A_{d,n}$ as varieties.*

The standard representation of SL_n on $\mathbb{C}^n$ induces an action of SL_n on $A_{d,n}$ for any d, and we have the following result (see [KRZ17]).

Proposition 1. *Let $n \geq 3$, and let Y be an affine normal variety of dimension n with a non-trivial SL_n-action. Then Y is SL_n -isomorphic to $A_{d,n}$ for some $d \geq 1$.*

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Now we drop the assumption of normality. Note that the ring of regular functions $\mathcal{O}(A_{d,n})$ equals $\bigoplus_{k=0}^{\infty} \mathbb{C}[x_1, ..., x_n]_{dk}$, where $\mathbb{C}[x_1, ..., x_n]_{dk}$ denotes the homogeneous polynomials of degree dk. Consider the affine variety $A_{d,n}^s$ with coordinate ring $\mathcal{O}(A_{d,n}^s) = \mathbb{C} \oplus \bigoplus_{k=s}^{\infty} \mathbb{C}[x_1, ..., x_n]_{dk} \subset \mathcal{O}(A_{d,n})$, $s \geq 1$. Then the induced morphism $\eta : A_{d,n} \to A_{d,n}^s$ is the normalization and has the property that the induced map $\eta' : A_{d,n} \setminus \{\star\} \xrightarrow{\sim} A_{d,n}^s \setminus \{\star\}$ is an isomorphism, where $\star$ denotes the points corresponding to the homogeneous maximal ideals. In fact, η is SL_n-equivariant, and $A_{d,n} \setminus \{\star\}$ is an SL_n-orbit. We prove the following result.

Theorem 2. *Let X be an irreducible affine variety such that $\mathrm{Aut}(X)$ and $\mathrm{Aut}(A_{d,n})$ are isomorphic as ind-groups, then $X \cong A_{d,n}^s$ as a variety for some $s \in \mathbb{N}$.*

For $n = 2$, any irreducible affine normal variety X endowed with a non-trivial SL_2-action is SL_2-isomorphic to $A_{d,2}$, SL_2/T or $\mathrm{SL}_2/N(T)$ (see [Pop73]), where T is the standard subtorus of SL_2 and $N(T)$ denotes the normalizer of T.

Theorem 3. *Let X be an irreducible variety such that $\mathrm{Aut}(X) \cong \mathrm{Aut}(\mathrm{SL}_2/T)$ respectively $\mathrm{Aut}(X) \cong \mathrm{Aut}(\mathrm{SL}_2/N(T))$ as ind-groups, then $X \cong \mathrm{SL}_2/T$ respectively $X \cong \mathrm{SL}_2/N(T)$ as varieties.*

For an affine variety X we denote by $U(X) \subset \mathrm{Aut}(X)$ the subgroup generated by the one-dimensional unipotent subgroups. We do not know whether $U(X)$ has the structure of an ind-subgroup (i.e. whether $U(X) \subset \mathrm{Aut}(X)$ is closed). That is why we introduce the definition of an "algebraic homomorphism". This is a homomorphism $\phi : U(X) \to U(Y)$ such that for any subgroup $U \subset U(X)$, where U is a closed one-dimensional unipotent subgroup of $\mathrm{Aut}(X)$, the image $\phi(U) \subset \mathrm{Aut}(Y)$ is a closed one-dimensional unipotent subgroup and $\phi|_U : U \to \phi(U)$ is an isomorphism of algebraic groups.

Theorem 4. *Let $n > 2$ and let X be an irreducible affine variety. If there is a bijective algebraic homomorphism $U(X) \to U(A_{d,n})$, then $X \cong A_{d,n}^s$ for some $s \geq 1$.*

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2. Preliminaries

The notion of an ind-group goes back to Shafarevich who called such objects *infinite dimensional groups*, (see [Sh66]). We refer to [Kum02] and [Kr15] for basic notions in this context.

Definition 1. By an *ind-variety* we mean a set V together with an ascending filtration $V_0 \subset V_1 \subset V_2 \subset ... \subset V$ such that the following holds:
(1) $V = \bigcup_{k \in \mathbb{N}} V_k$;
(2) each V_k has the structure of an algebraic variety;
(3) for all $k \in \mathbb{N}$ the subset $V_k \subset V_{k+1}$ is closed in the Zariski-topology.

A morphism from an ind-variety $V = \bigcup_k V_k$ to an ind-variety $W = \bigcup_m W_m$ is a map $\phi : V \to W$ such that for any k there is an m such that $\phi(V_k) \subset W_m$ and such that the induced map $V_k \to W_m$ is a morphism of algebraic varieties. *Isomorphisms* of ind-varieties are defined in the obvious way.

Two filtrations $V = \bigcup_{k\in N} V_k$ and $V = \bigcup_{k\in N} V'_k$ are called *equivalent* if for every k there is an m such that $V_k \subset V'_m$ is a closed subvariety as well as $V'_k \subset V_m$. Equivalently, the identity map $\mathrm{id} : V = \bigcup_{k\in N} V_k \to V = \bigcup_{k\in N} V'_k$ is an isomorphism of ind-varieties.

An ind-variety V has a natural topology: a subset $S \subset V$ is open, (resp. closed), if $S_k := S \cap V_k \subset V_k$ is open, (resp. closed), for all k. Naturally, a locally closed subset $S \subset V$ has a natural structure of an ind-variety. It is called an *ind-subvariety*. An ind-variety V is called *affine* if all varieties V_k are affine. Throughout this paper we consider only affine ind-varieties and for simplicity we call them just ind-varieties.

The product of two ind-varieties is defined in the natural way. This allows to give the following definition.

Definition 2. An ind-variety G is said to be an *ind-group* if the underlying set G is a group such that the map $G \times G \to G$, $(g, h) \mapsto gh^{-1}$, is a morphism.

An ind-group G is called *connected* if for every $g \in G$ there is an irreducible curve C and a morphism $C \to G$ whose image contains the neutral element e and g.

A closed subgroup H of G (i.e. H is a subgroup of G and is a closed subset) is again an ind-group under the closed ind-subvariety structure on G. A closed subgroup H of an ind-group G is an algebraic group if and only if H is an algebraic subset of G.

The proof of the next result can be found in [St13] (see also [FK17]).

Proposition 2. *Let X be an affine variety. Then* $\mathrm{Aut}(X)$ *has a natural structure of an affine ind-group.*

Note that in [St13] one can also find the description of the ind-group structure on $\mathrm{Aut}(X)$.

3. Automorphisms

Proposition 3. *Any automorphism of $A_{d,n}$ lifts to an automorphism of $\mathbb{C}^n$.*

Proof. Let $\phi \in \mathrm{Aut}(A_{d,n})$. First we claim that $p_i := \phi^*(x_i^d)$ and $p_j := \phi^*(x_j^d)$ are coprime in $\mathbb{C}[x_1, ..., x_n]$, where $i \neq j$ and ϕ^* is the pull-back of ϕ. Let p be a common factor of p_i and p_j. Then $\tilde{p} := \prod_{g\in\mu_d} gp$ divides p_i^d and p_j^d. By construction it is clear that $\tilde{p} \in \mathcal{O}(A_{d,n})$, then $\phi^{-1}(\tilde{p})$ is a common factor of $(\phi^*)^{-1}(p_i^d) = x_i^{d^2}$ and $(\phi^*)^{-1}(p_j^d) = x_j^{d^2}$. Hence, $\tilde{p} \in \mathbb{C}$ and therefore, $p \in \mathbb{C}$.

We have $\phi^*(x_i^d)\phi^*((x_j^d)^{d-1}) = \phi^*(x_i^d x_j^{d(d-1)}) = \phi^*(x_i x_j^{d-1})^d$ i.e. $p_i p_j^{d-1} = q^d$ for some $q \in \mathcal{O}(A_{d,n})$. Because p_i is coprime with p_j, it follows that $p_i = q_i^d$ for some $q_i \in \mathbb{C}[x_1, ..., x_n]$.

The map ϕ induces an automorphism of $A_{d,n} \setminus \{\pi(0)\}$ and we call it also by ϕ. Recall that the quotient $\pi : \mathbb{A}^n \to A_{d,n}$ induces an étale covering $\tilde{\pi} : \mathbb{A}^n \setminus \{0\} \to A_{d,n} \setminus \{\pi(0)\}$. As $\mathbb{A}^n \setminus \{0\}$ is simply connected, it follows that every continous automorphism of $A_{d,n} \setminus \{\pi(0)\}$ can be lifted to a continous automorphism of $\mathbb{A}^n \setminus \{0\}$. Since both varieties are complex manifolds and the covering is étale, the lift of a holomorphic automorphism is also holomorphic. Thus, the automorphism ϕ of $A_{d,n} \setminus \{\pi(0)\}$ lifts to a holomorphic automorphism ψ of $\mathbb{A}^n \setminus \{0\}$. Now consider

$q_i := \psi^*(x_i)$. This is a holomorphic function on $\mathbb{A}^n \setminus \{0\}$ with the property that $q_i^d = \psi^*(x_i^d) = \phi^*(x_i^d)$ is a polynomial. It follows that the meromorphic function $r_i := \frac{q_i}{p_i}$ is holomorphic outside the zero set of p_i and satisfies $r_i^d = 1$. This implies that r_i is a constant, hence $q_i = \omega_i p_i$ for some d-th root of unity ω_i, first outside the zero set of p_i and then everywhere. Thus $\psi^*(\mathbb{C}[x_1, ..., x_n]) \subset \mathbb{C}[x_1, ..., x_n]$ which means that ψ is an algebraic morphism $\mathbb{A}^n \to \mathbb{A}^n$. It is an isomorphism because ψ is bijective. □

Let X be an affine variety, H be a finite group acting on X and let $\pi : X \to X/H$ be the quotient morphism. Denote by $\mathrm{Aut}^H(X) \subset \mathrm{Aut}(X)$ the subgroup of all automorphisms of X which commute with the image of H in $\mathrm{Aut}(X)$.

Lemma 1. *(a)* $\mathrm{Aut}^H(X) \subset \mathrm{Aut}(X)$ *is a closed ind-subgroup,*

(b) there is a canonical homomorphism of ind-groups $\phi : \mathrm{Aut}^H(X) \to \mathrm{Aut}(X/H)$*,*

(c) if X is normal and contains only finitely many fixed points of H then every $\mathbb{C}^+$-action on X/H lifts to a $\mathbb{C}^+$-action on X.

Proof. (a) Consider the homomorphisms $\phi_h : \mathrm{Aut}(X) \to \mathrm{Aut}(X)$, $\phi_h(g) = ghg^{-1}$. Then $\mathrm{Aut}^H(X) = \cap_{h\in H}\phi_h^{-1}(H)$, where $\phi_h^{-1}(H) \subset \mathrm{Aut}(X)$ is a closed subvariety. This proves the claim.

(b) Now let $h \in H$, $f \in \mathcal{O}(X)^H$ and $\phi \in \mathrm{Aut}^H(X)$. Then $\phi^* : \mathcal{O}(X) \xrightarrow{\sim} \mathcal{O}(X)$ is an isomorphism and $h(\phi^*(f)) = \phi^*((\phi^*)^{-1} \circ h \circ \phi^*)(f) = (\phi^* \circ h')(f) = \phi^*(f)$ for some $h' \in H$. Therefore $\phi^*(f) \in \mathcal{O}(X)^H$, which means that ϕ induces an automorphism of X/H.

(c) There is an isomorphism of the space of derivations $\mathrm{Der}(\mathcal{O}(X))$ with $\mathrm{Hom}(\Omega^1_X, \mathcal{O}(X))$, where Ω^1_X denotes the Kähler differential forms on X. By [Ha80, Corollary 1.2], $\mathrm{Hom}(\Omega^1_X, \mathcal{O}(X))$ is a reflexive sheaf. Hence, $\mathrm{Hom}(\Omega^1_{X\setminus Y}, \mathcal{O}(X \setminus Y))$ coincides with $\mathrm{Hom}(\Omega^1_X, \mathcal{O}(X))$ for any closed subset $Y \subset X$ of codimension at least 2 (see [Ha80, Proposition 1.6]). Since X is normal, the quotient X/H is normal too. This implies that $\mathrm{Der}(\mathcal{O}(X/H)) = \mathrm{Der}(\mathcal{O}(X/H \setminus Z))$ for any closed subset $Z \subset X/H$ such that $\mathrm{codim}_{X/H}(Z) \geq 2$.

Let $Z \subset X/H$ be the image of the union of the set of fixed points under the action of the group H and the set of singular points of X. The map $\pi|_{X\setminus\pi^{-1}(Z)} : X \setminus \pi^{-1}(Z) \to X/H \setminus Z$ is a finite étale covering with group H. Hence, the pullback $\pi^*(T_{X/H\setminus Z})$ of the tangent bundle $T_{X/H\setminus Z}$ of $X/H\setminus Z$ coincides with $T_{X\setminus\pi^{-1}(Z)}$ and then $T_{X/H\setminus Z} = \pi_*^H(T_{X\setminus\pi^{-1}(Z)})$ which consists of H-invariant sections $X\setminus\pi^{-1}(Z) \to T_{X\setminus\pi^{-1}(Z)}$. This implies that $\mathrm{Der}(\mathcal{O}(X/H)) = \mathrm{Der}(\mathcal{O}(X/H\setminus Z))$ is naturally isomorphic to $\mathrm{Der}^H(\mathcal{O}(X \setminus \pi^{-1}(Z))) = \mathrm{Der}^H(\mathcal{O}(X))$, where $\mathrm{Der}^H(\mathcal{O}(X)) \subset \mathrm{Der}(\mathcal{O}(X))$ denotes the vector subspace of H-invariant derivations. This means that each derivation of $\mathcal{O}(X/H)$ lifts to a derivation of $\mathcal{O}(X)$ and then by [Vas69, Theorem 2.2], each locally nilpotent derivation of $\mathcal{O}(X/H)$ lifts to a locally nilpotent derivation of $\mathcal{O}(X)$. The claim follows. □

Let us recall that a closed subgroup U of $\mathrm{Aut}(X)$ is called a 1-dimensional unipotent subgroup if $U \cong \mathbb{C}^+$.

Proposition 4. *The homomorphism* $\phi_d : \mathrm{Aut}^{\mu_d}(\mathbb{A}^n) \to \mathrm{Aut}(A_{d,n})$ *is surjective with kernel* μ_d*. Moreover, every* 1*-dimensional unipotent subgroup of* $\mathrm{Aut}(A_{d,n})$ *is the image of some* 1*-dimensional unipotent subgroup of* $\mathrm{Aut}^{\mu_d}(\mathbb{A}^n)$*.*

Proof. The surjectivity of ϕ_d follows from Proposition 3. The last claim of the statement follows from Lemma 1 (c). What remains is to compute the kernel of ϕ_d.

It is clear that

$$\mathrm{Aut}^{\mu_d}(\mathbb{A}^n) = \{f = (f_1, ..., f_n) \in \mathrm{Aut}(\mathbb{A}^n) | f_i \in \bigoplus_{k=0}^{\infty} \mathbb{C}[x_1, ..., x_n]_{kd+1}, i = 1, ..., n\}.$$

Now let $f = (f_1, ..., f_n) \in \mathrm{Aut}^{\mu_d}(\mathbb{A}^n)$ be such that the map f' induced by f on $\mathbb{A}^n/\mu_d$ is the identity. This means that f' acts trivially on $\mathcal{O}(\mathbb{A}^n/\mu_d) = \mathbb{C} \oplus \bigoplus_{k\geq 1} \mathbb{C}[x_1, ..., x_n]_{kd}$. Hence, $f'(x_i^d) = x_i^d$ for any i which implies that $f = (\xi_1 x_1, ..., \xi_n x_n)$, where $\xi_i^d = 1$ for $i = 1, ..., n$. In particular, $f'(x_i^{d-1} x_j) = x_i^{d-1} x_j$ which implies that $\xi_i^{d-1}\xi_j = 1$ for any i, j. Because $\xi_i^{d-1}\xi_i = 1$ we conclude that $\xi_i = \xi_j$. The claim follows. □

4. Root subgroups

Let G be an ind-group, and let $T \subset G$ be a closed torus.

Definition 3. A closed subgroup $U \subset G$ isomorphic to $\mathbb{C}^+$ and normalized by T is called a root subgroup with respect to T. The character of T on $\mathrm{Lie}\, U \cong \mathbb{C}$ i.e. the algebraic action of T on $\mathrm{Lie}\, U$ is called the weight of U.

Let X be an affine variety and consider a nontrivial algebraic action of $\mathbb{C}^+$ on X, given by $\lambda : \mathbb{C}^+ \to \mathrm{Aut}(X)$. If $f \in \mathcal{O}(X)$ is $\mathbb{C}^+$-invariant, then the *modification* $f \cdot \lambda$ of λ is defined in the following way:

$$(f \cdot \lambda)(s)x := \lambda(f(x)s)x$$

for $s \in \mathbb{C}$ and $x \in X$. It is easy to see that this is again a $\mathbb{C}^+$-action. In fact, the corresponding locally nilpotent derivation to $f \cdot \lambda$ is $f\delta_\lambda$, where δ_λ is the locally nilpotent derivation which correspond to λ. It is clear that if X is irreducible and $f \neq 0$, then $f \cdot \lambda$ and λ have the same invariants. If $U \subset \mathrm{Aut}(X)$ is a closed subgroup isomorphic to $\mathbb{C}^+$ and if $f \in \mathcal{O}(X)^U$ is a U-invariant, then in a similar way we define the modification $f \cdot U$ of U. Choose an isomorphism $\lambda : \mathbb{C}^+ \to U$ and set $f \cdot U := \{(f \cdot \lambda)(s) | s \in \mathbb{C}^+\}$. Note that $\mathrm{Lie}(f \cdot U) = f\, \mathrm{Lie}\, U \subset \mathrm{Vec}(X)$.

If a torus T acts linearly and rationally on a vector space V, then we call V multiplicity-free if the weight spaces V_α are all of dimension ≤ 1.

Lemma 2 ([Kr15])**.** *Let X be an irreducible affine variety and let $T \subset \mathrm{Aut}(X)$ be a torus. Assume that there exists a root subgroup $U \subset \mathrm{Aut}(X)$ with respect to T such that the T-module $\mathcal{O}(X)^U$ is multiplicity-free. Then $\dim T \leq \dim X \leq \dim T + 1$.*

5. A special subgroup of $\mathrm{Aut}(X)$

For any affine variety X consider the normal subgroup $U(X)$ of $\mathrm{Aut}(X)$ generated by closed one-dimensional unipotent subgroups. The group $U(X)$ was introduced and studied in [AFK13], where the authors called it the group of special automorphisms of X. After [Kr15] we introduce the following notion of an algebraic homomorphism between these groups.

Definition 4. A homomorphism $\phi : U(X) \to U(Y)$ is algebraic if for any subgroup $U \subset U(X)$ such that $U \subset \mathrm{Aut}(X)$ is closed, $U \cong \mathbb{C}^+$, the image $\phi(U) \subset \mathrm{Aut}(Y)$ is closed and $\phi|_U : U \to \phi(U)$ is a homomorphism of algebraic groups. We say that $U(X)$ and $U(Y)$ are algebraically isomorphic, $U(X) \cong U(Y)$, if there exists a bijective homomorphism $\phi : U(X) \to U(Y)$ such that ϕ and ϕ^{-1} are both algebraic.

A subgroup $G \subset U(X)$ is called algebraic if $G \subset \mathrm{Aut}(X)$ is the closed algebraic subgroup. The next lemma can be found in [Kr15, Lemma 4.2].

Lemma 3. *Let $\phi : U(X) \to U(Y)$ be an algebraic homomorphism. Then, for any algebraic subgroup $G \subset U(X)$ generated by one-dimensional unipotent subgroups of $\mathrm{Aut}(X)$, the image $\phi(G)$ is an algebraic subgroup of $U(Y)$ and $\phi|_G : G \to \phi(G)$ is a homomorphism of algebraic groups.*

Lemma 4. *Let X be an irreducible affine variety, and let $\eta\colon \tilde{X} \to X$ be its normalization. Then every automorphism of X lifts uniquely to an automorphism of $\tilde{X}$ and the induced map $\tilde{\eta} : U(X) \hookrightarrow U(\tilde{X})$ is an algebraic homomorphism.*

Proof. Let $\mathbb{C}(X)$ be the field of rational functions on X. Then any automorphism ϕ of the ring of regular functions $\mathcal{O}(X)$ uniquely extends to an automorphism $\phi^{'}$ of $\mathbb{C}(X)$. We claim that $\mathcal{O}(\tilde{X})$ is invariant under $\phi^{'}$, which would prove the first part of the lemma. Indeed, by definition f belongs to $\mathcal{O}(\tilde{X})$ if there is a monic polynomial $F = t^n + c_1 t^{n-1} + ... + c_n \in \mathcal{O}(X)[t]$ such that $F(f) = 0$. Then $\phi(F(f)) = G(\phi(f)) = 0$ for some monic $G \in \mathcal{O}(X)[t]$, which proves the claim.

To prove the second part of the lemma, we note that any action of an algebraic group G on X lifts uniquely to a G-action on $\tilde{X}$. This follows from the fact that $G \times \tilde{X}$ is normal, the universal property of normalization and the following diagram:

$$\begin{array}{ccc} G \times \tilde{X} & \longrightarrow & \tilde{X} \\ \Big\downarrow{\scriptstyle \mathrm{id}_G \times \eta} & & \Big\downarrow{\scriptstyle \eta} \\ G \times X & \longrightarrow & X \end{array}$$

Therefore, each regular $\mathbb{C}^+$-action on X lifts uniquely to a regular $\mathbb{C}^+$-action on $\tilde{X}$, which proves the claim. □

Proposition 5. *Let $n \geq 3$ and let X be an n-dimensional irreducible affine variety endowed with a non-trivial SL_n-action. Then $\mathcal{O}(X) = \mathbb{C} \oplus \sum_{i=1}^{l} \sum_{k=k_i}^{\infty} \mathbb{C}[x_1, ..., x_n]_{kd_i}$ for some $l, k_i, d_i \in \mathbb{N}$. The same holds when $n = 2$ and the normalization of X is $A_{d,2}$ for some $d \in \mathbb{N}$.*

Proof. First, let $n \geq 3$. If X is normal, then by Proposition 1, $X \cong A_{d,n}$ for some $d \in \mathbb{N}$. It is clear that $\mathcal{O}(A_{d,n}) = \bigoplus_{k=0}^{\infty} \mathbb{C}[x_1, ..., x_n]_{kd}$ is a direct sum of irreducible pairwise non-isomorphic SL_n-modules $\mathbb{C}[x_1, ..., x_n]_{kd}$.

Now, consider any n-dimensional irreducible affine variety X endowed with a non-trivial SL_n-action and a normalization morphism $\eta : A_{d,n} \to X$. Since any SL_n-action on $\mathcal{O}(X)$ lifts to an SL_n-action on $\mathcal{O}(A_{d,n})$, it follows that $\mathcal{O}(X)$ is a SL_n-submodule of $\mathcal{O}(A_{d,n})$ and therefore $\mathcal{O}(X) = \bigoplus_{k \in \Omega} \mathbb{C}[x_1, ..., x_n]_{kd}$, where Ω is a submonoid of $\mathbb{N}$ under addition. Since $\mathcal{O}(X)$ is finitely generated, $\Omega \subset \mathbb{N}$ is a finitely generated submonoid i.e. there exist $k_1, ..., k_l \in \mathbb{N}$ such that $\Omega = k_1\mathbb{N} + ...k_l\mathbb{N}$. The claim follows. □

6. 2-DIMENSIONAL CASE

The next result can be found in [Pop73], §3 (see also [Kr84], §4).

Lemma 5. *Let X be an affine normal irreducible variety of dimension two endowed with a non-trivial SL_2-action. Then X is SL_2-isomorphic to one of the following varieties:*

(a) $A_{d,2}$ *for some* $d \in \mathbb{N}$,
(b) SL_2/T, *where* T *is the standard subtorus of* SL_2,
(c) $\mathrm{SL}_2/N(T)$, *where* $N(T)$ *is the normalizer of* T.

The SL_2-variety $A_{d,2}$ is the union of a fixed point and the orbit $(\mathbb{C}^2 \setminus \{0\})/\mu_d \cong \mathrm{SL}_2/U_d$, where μ_d acts by scalar multiplication on $\mathbb{C}^2 \setminus \{0\}$ and $U_d = \{\begin{bmatrix} \xi & t \\ 0 & \xi^{-1} \end{bmatrix} | t \in \mathbb{C}, \xi \in \mathbb{C}^*, \xi^d = 1\}$. Moreover, any closed subgroup of SL_2 of codimension ≤ 2 is either T, $N(T)$, U_d for some $d \geq 1$ or $B = \{\begin{bmatrix} a & t \\ 0 & a^{-1} \end{bmatrix} | t \in \mathbb{C}, a \in \mathbb{C}^*\}$ (see [We52]).

The next result can be found in [Kr84, III.2.5, Folgerung 3].

Proposition 6. *If a reductive group* G *acts on an affine variety* X *and if the stabilizer of a point* $x \in X$ *contains a maximal torus, then the orbit* Gx *is closed.*

Proposition 7. *Let* X *be an* SL_2*-variety and let* $O = \mathrm{SL}_2 x$ *be the orbit of* x*. Assume that* $\dim O \leq 2$*. Then we are in one of the following cases:*
(a) x *is a fixed point;*
(b) the orbit O *is closed and* SL_2*-isomorphic to* SL_2/T *or* $\mathrm{SL}_2/N(T)$*;*
(c) $\overline{O} = O \cup \{x_0\}$*, where* $\overline{O}$ *is the closure of the orbit* O *and* x_0 *is a fixed point. Moreover, either* $\overline{O} \simeq \mathbb{A}^2$ *or* x_0 *is an isolated singular point.*

Proof. If the stabilizer of x contains a maximal torus then we are in case (a) or (b) by Proposition 6. Otherwise, from the classification of closed subgroups of SL_2 it follows that the stabilizer of x coincides with U_d for some $d \geq 1$ and $\overline{O}$ does not contain orbits of dimension one. Hence, $\overline{O} = O \cup \{x_0\}$. It is clear that if $\overline{O}$ is singular, then x_0 is an isolated singular point. If $\overline{O}$ is smooth, then from Lemma 5 it follows that $\overline{O}$ is isomorphic to $\mathbb{A}^2$. □

Note that $\mathrm{SL}_2/T \cong \mathbb{P}^1 \times \mathbb{P}^1 \setminus \Delta$, where Δ is the diagonal, and $\mathrm{SL}_2/N(T) \cong \mathbb{P}^2 \setminus C$, where C is a smooth conic (see [Pop73, Lemma 2]).

There is the following well-known result.

Lemma 6. *Let* X *be a variety and let* $G \subset \mathrm{Aut}(X)$ *be an algebraic subgroup. Assume that* $X = Gx$ *for* $x \in X$*. Then* $\mathrm{Aut}^G(X) \cong N_G(Gx)/G_x$*.*

In fact, the right-multiplications on G/H with elements from $N_G(H)/H$ are the automorphisms of G/H which commute with the left-multiplications with all elements from G.

Lemma 7. *Consider the natural* SL_2*-action on* $X = \mathrm{SL}_2/T, \mathrm{SL}_2/N$ *or* $A_{d,2}$*, and denote by* $S \subset \mathrm{Aut}(X)$ *the image of* SL_2*.*

(a) If $X = \mathrm{SL}_2/T$*, then* $S \cong \mathrm{PSL}_2$ *and* $\mathrm{Aut}^S(X) = \{\tau, \mathrm{id}\}$*. Moreover,* τ *acts freely on* X*, and* $X/\tau \cong \mathrm{SL}_2/N(T)$*.*

(b) If $X = \mathrm{SL}_2/N(T)$*, then* $S \cong \mathrm{PSL}_2$ *and* $\mathrm{Aut}^S(X) = \mathrm{id}$*.*

(c) If $X = A_{d,2}$*, then* $S \cong \mathrm{SL}_2$ *if* d *is odd and* $S \cong \mathrm{PSL}_2$ *if* d *is even. Moreover,* $\mathrm{Aut}^S(X) \cong \mathbb{C}^*$ *is given by the image of* $\mathbb{C}^*$ *acting by scalar multiplication on* $\mathbb{A}^2$*. In particular, the groups* $\mathrm{Aut}(\mathrm{SL}_2/T)$ *and* $\mathrm{Aut}(\mathrm{SL}_2/N(T))$ *are not isomorphic, and also not isomorphic to* $\mathrm{Aut}(A_{d,2})$ *for any* $d \geq 1$*.*

Proof. Since the natural action of SL_2 on SL_2/T or $\mathrm{SL}_2/N(T)$ is transitive, (a) and (b) are immediate consequences of Lemma 6. For (c) we remark that X contains the orbit $O \cong \mathrm{SL}_2/U_d$. For $d = 1$, i.e. for $X = \mathbb{A}^2$, the claim is well-known. If $d > 1$, then $\mathrm{Aut}(X) \cong \mathrm{Aut}(O)$, since the complement of O in X is a singular point. Now the claim follows from Lemma 6. □

The variety SL_2 / T is isomorphic to the following so-called DANIELEWSKI surface, i.e. the smooth 2-dimensional affine quadric $V(xz - y^2 + y) \subset \mathbb{A}^3$ (see [DP09]) and the quotient map $\pi : \mathrm{SL}_2 \to \mathrm{SL}_2 / T$ is given by $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \to (ab, ad, cd)$. It is not difficult to see that $X := V(xz + y^2 - 1) \cong V(xz - y^2 + y) \subset \mathbb{A}^3$.

By Lemma 7, there is an automorphism $\tau \in \mathrm{Aut}^S(X)$ which acts freely on X and the quotient $Y := X/\tau$ is isomorphic to $\mathrm{SL}_2 / N(T)$, i.e. $\pi : X \to Y$ is a principal $\mathbb{Z}/2$-bundle. In particular, $\mathcal{O}(Y) = \mathcal{O}(X)^\tau$. An automorphism ϕ of X descends to an automorphism on Y if and only if ϕ sends τ-orbits to τ-orbits. In fact, such an automorphism sends τ -invariant functions of $\mathcal{O}(X)$ to τ-invariant functions of $\mathcal{O}(X)$. Since τ has order 2, this condition for ϕ is equivalent to the condition that ϕ commutes with τ. We first note that $\mathrm{Aut}^\tau(X)$ is a closed subgroup of $\mathrm{Aut}(X)$ and then the canonical map $p : \mathrm{Aut}^\tau(X) \to \mathrm{Aut}(Y)$ is a homomorphism of ind-groups. In fact, kernel of p equals $\langle \tau \rangle$.

The following proposition follows from Lemma 1(c).

Proposition 8. *Every $\mathbb{C}^+$-action on Y lifts to a $\mathbb{C}^+$-action on X. In particular, the image $p(\mathrm{Aut}_\tau(X))$ contains $U(Y)$ and $p^{-1}(U(Y)) \subset U(X)$*

Corollary 1. *For every algebraic subgroup $G \subset U(Y)$ the inverse image $\pi^{-1}(G) \subset \mathrm{Aut}_\tau(X)$ is algebraic. If G is generated by unipotent elements, then $\pi^{-1}(G) = \pi^{-1}(G)^0 \times \langle \tau \rangle$.*

By [Lam05, Theorem 6], $\mathrm{Aut}(X)$ is the amalgamated product of the orthogonal group $\mathrm{O}(3, \mathbb{C}) = \mathrm{SO}(3, \mathbb{C}) \times \langle \tau \rangle$ and $J_T \rtimes \langle \tau \rangle$ along their intersection C_T, where $\tau = (-x, -y, -z)$, J_T is the group of automorphisms of the form

$$(x, y, z) \mapsto (\alpha x + 2\alpha y P(z) - \alpha z P^2(z), (y - zP(z)), \frac{1}{\alpha} z);\ \alpha \in \mathbb{C}^*, P \in \mathbb{C}[z].$$

Hence, $\mathrm{Aut}(X)$ is generated by $U(X)$ and $\langle \tau \rangle$. Since $U(X)$ is the normal subgroup of $\mathrm{Aut}(X)$, it follows that $\mathrm{Aut}(X) = U(X) \rtimes \langle \tau \rangle$. By [Neu48, Corollary 8.11], $U(X)$ is the amalgamated product of $\mathrm{SO}(3, \mathbb{C})$ and J_T. Note that the subgroup $U(X) = \mathrm{Aut}^0(X) \subset \mathrm{Aut}(X)$ is closed (see [Kr15, Lemma 6.3]), where $\mathrm{Aut}^0(X)$ is the neutral component of $\mathrm{Aut}(X)$. Hence, $U(X)$ is an ind-group. By the following computation

$$\begin{aligned}(tx, y, t^{-1}z) \circ (x + 2yP(z) - zP^2(z), (y - zP(z)), z) \circ (t^{-1}x, y, tz) = \\ = (x + 2ytP(tz) - zt^2P^2(tz), (y - ztP(tz)), z),\end{aligned}$$

it is easy to see that $U_i = \{(x + 2yP_i(z) - zP_i^2(z), (y - zP_i(z)), z) | P_i(z) = z^i\}$ is the root subgroup with weight $i + 1$ with respect to $T'' = \{(tx, y, t^{-1}z) | t \in \mathbb{C}^*\} \cong \mathbb{C}^*$ for any $i \in \mathbb{N} \cup \{0\}$. The fact that there is no other root subgroups with respect to T'' follows from amalgamated product structure.

Summarizing everything that is said above, we have the following result.

Proposition 9. *For $X = \mathrm{SL}_2 / T$ we have the following properties.*

(a) *All closed subgroups $S \subset \mathrm{Aut}(X)$ isomorphic to PSL_2 are conjugate.*

(b) *The root subgroups with respect to a maximal torus T'' of some $S \cong \mathrm{PSL}_2$ are multiplicity-free with weights $1, 2, 3, ...$*

It is not difficult to see that $\mathrm{Aut}^{\tau}(X)$ is the amalgamated product of $\mathrm{SO}(3,\mathbb{C}) \times \langle\tau\rangle$ and $J^{\tau} \times \langle\tau\rangle$ along their intersection, where

$$J^{\tau} = \{(x,y,z) \mapsto (\alpha x + 2\alpha y P(z) - \alpha z P^2(z), (y - zP(z)), \frac{1}{\alpha} z);\ \alpha \in \mathbb{C}^*, P \in \bigoplus_{l=0}^{\infty} \mathbb{C} z^{2l}\}.$$

By [Neu48, Corollary 8.11], $\mathrm{Aut}^{\tau}(X)$ is the amalgamated product of $\mathrm{SO}(3,\mathbb{C}) \times \langle\tau\rangle$ and $J^{\tau} \times \langle\tau\rangle$ along their intersection.

Recall that map $p : \mathrm{Aut}^{\tau}(X) \twoheadrightarrow \mathrm{Aut}(Y)$ is the surjective homomorphism with kernel $\langle\tau\rangle$. Hence, $\mathrm{Aut}(Y) = \mathrm{Aut}^{\tau}(X)/\langle\tau\rangle$. By [Co63, Theorem 1], $\mathrm{Aut}(Y)$ is the amalgamated product of $\mathrm{SO}(3,\mathbb{C})$ and J^{τ} along their intersection. Therefore, $\mathrm{Aut}(Y) = U(Y)$.

Summarizing everything that is said above and Proposition 9, we have the following result.

Corollary 2. *The root subgroups with respect to a maximal torus T'' of any $S \cong \mathrm{PSL}_2$ are multiplicity-free with weights 1,3,5,.... In particular, $U(\mathrm{SL}_2/N(T)) \ncong U(\mathrm{SL}_2/T)$.*

Recall that by Corollary 4, there is a homomorphism $\phi_d : \mathrm{Aut}^{\mu_d}(\mathbb{A}^n) \to \mathrm{Aut}(A_{d,n})$ of ind-groups. Consider now the torus $T_n = \{(t_1,...,t_n) | t_i \in \mathbb{C}^*\} \subset \mathrm{Aut}(\mathbb{A}^n)$ and the torus $T_n' = \{(t_1,...,t_n) | t_i \in \mathbb{C}^*, t_1 \cdot ... \cdot t_n = 1\} \subset U(\mathbb{A}^n)$ of dimension $n-1$. Then $T_d := \phi_d(T_n')$ is a maximal subtorus of $U(A_{d,n}) \subset \mathrm{Aut}(A_{d,n})$.

The following lemma is easy and follows from Lemma 12.

Lemma 8. *Let d be even. Then weights of root subgroups of $\mathrm{Aut}(A_{d,2})$ with respect to T_d are $\{\frac{kd+2}{2} |\ k \in \mathbb{N} \cup \{0\}\}$.*

By Jung - Van der Kulk theorem (see [Ju42] and [Kul53]) $\mathrm{Aut}(\mathbb{A}^2) = \mathrm{Aff}_2 *_C J$, where Aff_2 is the group of affine transformations of $\mathbb{A}^2$, $J = \{(ax+b, cy+f(x)) | a,c \in \mathbb{C}^*, b \in \mathbb{C}, f(y) \in \mathbb{C}[x]\}$ and $C = \mathrm{Aff}_2 \cap J$. Subgroup $\mathrm{Aut}^{\mu_k}(\mathbb{A}^2) \subset \mathrm{Aut}(\mathbb{A}^2)$ also has a structure of amalgamated product by [Neu48, Corollary 8.11], namely, $\mathrm{Aut}^{\mu_k}(\mathbb{A}^2)$ is the amalgamated product of GL_2 and $J_k = \{(ax+b, cy+f(x)) | a,c \in \mathbb{C}^*, b \in \mathbb{C}, f(y) \in \bigoplus_l \mathbb{C}x^{lk+1}\}$ along their intersection (see also [AZ13, Theorem 4.2]). From Proposition 4, it follows that $\mathrm{Aut}(A_{k,2}) \cong \mathrm{Aut}^{\mu_k}(\mathbb{A}^2)/\mu_k$ and by [Co63, Theorem 1], $\mathrm{Aut}^{\mu_k}(\mathbb{A}^2)/\mu_k$ is the amalgamated product of GL_2/μ_k and J_k/μ_k along their intersection C_k. Hence, it is easy to see that $U(\mathbb{A}^2/\mu_{2k})$ is the amalgamated product of PSL_2 and $J_{2k} = \{(ax+b, cy+f(x)) | a,c \in \mathbb{C}^*, b \in \mathbb{C}, f(y) \in \bigoplus_l \mathbb{C}x^{lk+1}\}$ along their intersection.

Note that $\mathcal{O}(A_{d,n}) \subset \mathbb{C}[x_1,...,x_n]$ for any $d \geq 1$. Hence, we can define the Jacobian matrix of $f = (f_1,...,f_n) \in \mathrm{Aut}(A_{d,n})$ in the ususal way i.e. $\mathrm{Jac}(f) = (\frac{\partial f_i}{\partial x_j})_{i,j}$ and then $\jmath(f) := \det \mathrm{Jac}(f)$. It is also well-known that $U(\mathbb{A}^2) = \{f \in \mathrm{Aut}(\mathbb{C}^2) |\ \jmath(f) \in \mathbb{C}^*\}$. It follows that $U(A_{d,2}) = \{f \in \mathrm{Aut}(A_{d,2}) |\ \jmath(f) \in \mathbb{C}^*\}$. Therefore, $U(A_{d,2}) \subset \mathrm{Aut}(A_{d,2})$ is the closed subgroup.

The following result was pointed to us by Hanspeter Kraft.

Proposition 10. *Let Z be an irreducible affine normal variety of dimension 2.*

(a) Assume that there is a bijective algebraic homomorphism $U(\mathrm{SL}_2/T) \to U(Z)$. Then $Z \cong \mathrm{SL}_2/T$ or $A_{2,2}$.

(b) Assume that there is a bijective algebraic homomorphism $U(\mathrm{SL}_2/N(T)) \to U(Z)$. Then $Z \cong \mathrm{SL}_2/N(T)$ or $A_{4,2}$.

Proof. Choose an SL_2-action on Z such that the the root subgroups with respect to the image $T \subset U(Z)$ of the diagonal torus $T \subset \mathrm{SL}_2$ are multiplicity-free with weights $1, 2, 3, \ldots$. The existence of such an action is given by Proposition 9(b) for SL_2/T , and then follows for $\mathrm{SL}_2/N(T)$ by Corollary 2. By Lemma 5, Z is SL_2-isomorphic to SL_2/T, to $\mathrm{SL}_2/N(T)$, or to $A_{d,2}$ for some $d \in \mathbb{N}$.

To prove the claim, we first note that $U(\mathrm{SL}_2/T) \ncong U(\mathrm{SL}_2/N(T))$ by Corollary 2. Let $X \cong \mathrm{SL}_2/T$ or to $\mathrm{SL}_2/N(T)$. Then the isomorphism $U(X) \cong U(\mathbb{C}^2/\mu_d)$ implies that d is even by Lemma 7. By Lemma 13, weights of root subgroups of $U(X)$ and $U(A_{d,2})$ have to be equal and then Lemma 8 implies that $U(\mathrm{SL}_2/T)$ can only be isomorphic to $U(A_{2,2})$, and $U(\mathrm{SL}_2/N(T))$ can only be isomorphic to $U(A_{4,2})$ by Corollary 2.

To show that $U(A_{2,2})$ and $U(\mathrm{SL}_2/T)$ are algebraically isomorphic, we first note that the first factors from the amalgamated product (described above) of $U(A_{2,2})$ and $U(\mathrm{SL}_2/T)$ are isomorphic to PSL_2. To show that J_2 and J_T are algebraically isomorphic, it is enough to say that they have the same weights with respect to the standart subtori. It remains to remark that $C_T \cong C_2$. Analogously, $U(A_{4,2})$ and $U(\mathrm{SL}_2/N(T))$ are algebraically isomorphic too. □

7. Higher-dimensional case

The next result can be found in [Lie11, Theorem 1]. Recall that by T'_n we denote the standard maximal subtorus of $\mathrm{SAut}(\mathbb{A}^n) = \{f = (f_1, \ldots, f_n) \in \mathrm{Aut}(\mathbb{A}^n) | \operatorname{jac}(f) := \det[\frac{\partial f_i}{\partial x_j}]_{i,j} = 1\}$.

Lemma 9. *Let $U \subset \mathrm{SAut}(\mathbb{A}^n)$ be a one-dimensional unipotent subgroup. Then U is a root subgroup with respect to T'_n if and only if $U = U_\lambda = \{(x_1, \ldots, x_i + cm_i, \ldots, x_n) | c \in \mathbb{C}\}$, where $m_i = x_1^{\lambda_1} \ldots x_{i-1}^{\lambda_{i-1}} x_{i+1}^{\lambda_{i+1}} \ldots x_n^{\lambda_n}$. The character ξ_λ corresponding to the root subgroup U is the following: $\xi_\lambda : T'_n \to \mathbb{C}^*$, $t = (t_1, \ldots, t_n) \mapsto t_i t_1^{-\lambda_1} \ldots \hat{t}_i \ldots t_n^{-\lambda_n}$.*

Remark 1. The last lemma can also be expressed in the following way (see [KS13, Remark 2]): there is a bijective correspondence between the T'_n-stable one-dimensional unipotent subgroups $U \subset \mathrm{Aut}(\mathbb{A}^n)$ and the characters of T'_n of the form $\lambda = \sum_j \lambda_j \epsilon_j$ where one λ_i equals 1 and the others are ≤ 0. We will denote this set of characters by $X_u(T'_n)$:

$$X_u(T'_n) := \{\lambda = \sum \lambda_j \epsilon_j | \textit{such that } \lambda_i = 1 \textit{ and } \lambda_j \leq 0 \textit{ for } j \neq i\}.$$

If $\lambda \in X_u(T'_n)$, then U_λ denotes the corresponding one-dimensional unipotent subgroup normalized by T'_n.

Lemma 10. *Consider the standard action of SL_n on $A_{d,l}$ and denote by $S_{n,d} \subset \mathrm{Aut}(A_{d,n})$ the image of SL_n. Then $S_{n,d} \cong \mathrm{SL}_n/\mu_{(n,d)}$, where (n, d) denotes the greatest common divisor of n and d. Moreover, $S_{n,d} \subset U(A_{d,n})$.*

Proof. By Proposition 4, there is a surjective homomorphism $\phi_d : \mathrm{Aut}^{\mu_d}(\mathbb{A}^n) \to \mathrm{Aut}(A_{d,n})$ of ind-groups with $\ker \phi = \mu_d$. Hence, $\mathrm{Aut}(A_{d,n}) \cong \mathrm{Aut}^{\mu_d}(\mathbb{A}^n)/\mu_d$ which shows that $S_{n,d} \cong \mathrm{SL}_n/\mu_d$. The second claim is clear. □

Corollary 3. *If $U(A_{d,n})$ and $U(A_{l,n})$ are algebraically isomorphic, then $(d, n) = (l, n)$.*

Recall that by Proposition 4 there is a homomorphism $\phi_d : \mathrm{Aut}^{\mu_d}(\mathbb{A}^n) \to \mathrm{Aut}(A_{d,n})$ of ind-groups and we denote by T_d the subtorus $\phi_d(T_n') \subset U(A_{d,n})$. Map ϕ_d induces the map $\tilde{\phi}_d : U^{\mu_d}(\mathbb{A}^n) \to U(A_{d,n})$ which has the kernel $\mu_{(n,d)}$, where $U^{\mu_d}(\mathbb{A}^n) \subset \mathrm{Aut}^{\mu_d}(\mathbb{A}^n)$ is a subgroup generated by $\mathbb{C}^+$-actions.

In [BB67], it is proved that any faithful action of an $(n-1)$-dimensional torus on $\mathbb{A}^n$ is linear. This result is used in order to prove the following lemma.

Lemma 11. *Let T be an algebraic subtorus of $U(A_{d,n})$ of dimension $(n-1)$. Then there exists a bijective algebraic homomorphism $F : U(A_{d,n}) \xrightarrow{\sim} U(A_{d,n})$ such that $F(T) = T_d$.*

Proof. Torus $(\phi_d^{-1}(T))^0$ is an algebraic subgroup of $U(\mathbb{A}^n)$ isomorphic to $(\mathbb{C}^*)^{n-1}$. By [BB67, Theorem 1], the torus $\phi_d^{-1}(T)^\circ$ is conjugate to some subtorus $\tilde{T}$ of T_n in $\mathrm{Aut}(\mathbb{A}^n)$. Since $U(\mathbb{A}^n)$ is the normal subgroup of $\mathrm{Aut}(\mathbb{A}^n)$, $\tilde{T} \subset T_n' = T_n \cap U(\mathbb{A}^n)$. Therefore, $(\phi_d^{-1}(T))^0$ is conjugate to T_n' which proves the claim. $\square$

Lemma 12. *Let $U \subset \mathrm{Aut}(A_{d,n})$ be a root subgroup with respect to T_d which has a character χ. Then U lifts to a root subgroup $\tilde{U} := (\phi_d^{-1}(U))^0 \subset \mathrm{Aut}_{\mu_d}(\mathbb{A}^n)$ with respect to $T_n' = (\phi_d^{-1}(T_d))^0$ with character $\tilde{\chi} := \psi^*(\chi)$ such that the following diagram*

$$\begin{tikzcd} 1 \arrow[r] & \mu_{(n,d)} \arrow[r] & T_n' \arrow[r, "\psi"] \arrow[d, "\tilde{\chi}"] & T_d \arrow[r] \arrow[d, "\chi"] & 1 \\ & & \mathbb{C}^* \arrow[r, "="] & \mathbb{C}^* & \end{tikzcd}$$

commute, where $\psi = \phi_d|_{T_n'}$ and $\psi^*(\chi)$ is a pull-back of χ.

Proof. From Proposition 3 it follows that any root subgroup U of $\mathrm{Aut}(A_{d,n})$ with respect to T_d lifts to a unipotent subgroup $\tilde{U} = (\phi_d^{-1}(U))^0$ of $\mathrm{Aut}^{\mu_d}(\mathbb{A}^n)$. Moreover, $\tilde{U}$ is normalized by $(\phi_d^{-1}(T_d))^\circ = T_n'$. Now, let $\tilde{u} \in \tilde{U}$ and $u = \phi_d(\tilde{u}) \in U$. Then $\phi_d(t^{-1} \circ \tilde{u}(s) \circ t) = \phi_d(\tilde{u}(t^k s)) = \tilde{u}(\psi(t^k)s)$ for some $k \in \mathbb{N}$, which proves the claim. $\square$

Proposition 11. *Let $X = A_{d,n}$, SL_2/T or $\mathrm{SL}_2/N(T)$ and Y be an irreducible affine variety. Let also assume that there is a bijective algebraic homomorphism $U(X) \xrightarrow{\sim} U(Y)$. Then $\dim Y \leq \dim X$. Moreover, if additionally Y is normal, then*

(a) if $X \cong \mathrm{SL}_2/T$, then $Y \cong A_{2,2}$ or $Y \cong \mathrm{SL}_2/T$,
(b) if $X \cong A_{2,2}$, then $Y \cong A_{2,2}$ or $Y \cong \mathrm{SL}_2/T$,
(c) if $X \cong \mathrm{SL}_2/N(T)$, then $Y \cong A_{4,2}$ or $Y \cong \mathrm{SL}_2/N(T)$,
(d) if $X \cong A_{4,2}$, then $Y \cong A_{4,2}$ or $Y \cong \mathrm{SL}_2/N(T)$,
(e) otherwise, $Y \cong A_{d,n}$.

Proof. Fix an algebraic isomorphism $\psi : U(X) \xrightarrow{\sim} U(Y)$ and denote by T' the image of T_d if $X = A_{d,2}$ or the image of a maximal subtorus T of $U(X)$ if $X = \mathrm{SL}_2/T$ or $\mathrm{SL}_2/N(T)$. By Lemma 12, Proposition 9 and Corollary 2, all root subgroups $U \subset U(Y)$ with respect to T' have different weights. In particular, the root subgroups $\mathcal{O}(Y)^U \cdot U \subset U(Y)$ have different weights, which implies that $\mathcal{O}(Y)^U$ is multiplicity-free, because the map $\mathcal{O}(Y)^U \to \mathcal{O}(Y)^U \cdot U$ is injective. Hence, by Lemma 2, we have that $\dim Y \leq \dim T' + 1 = n$, which proves the first part of the lemma.

Now (a), (b), (c) and (d) follow from Proposition 10.

To prove (e), we note that $\mathrm{SL}_n/\mu_{(n,d)}$ belongs to $U(A_{d,n})$, which implies that SL_n acts non-trivially on Y and thus, by Proposition 1, $Y \cong A_{l,n}$ for some $l \in \mathbb{N}$. Hence, $\psi : U(A_{d,n}) \xrightarrow{\sim} U(A_{l,n})$. By Lemma 11 there exist an algebraic isomorphism $F : U(A_{l,n}) \xrightarrow{\sim} U(A_{l,n})$ such that $F(\psi(T_d)) = T_l$. Therefore, we can assume that $\psi(T_d) = T_l$. Groups $U(A_{l,n})$ and $U(A_{d,n})$ can be isomorphic only if $(n,d) = (n,l)$ by Corollary 3. Then, by Lemma 13, weights of root subgroups of $U(A_{d,n})$ and $U(A_{l,n})$ with respect to tori T_d and T_l respectively have to coincide and the claim follows from Lemma 12. □

Proof of Theorem 1. It is clear from the definition that an isomorphism of ind-groups $\mathrm{Aut}(X) \xrightarrow{\sim} \mathrm{Aut}(A_{d,n})$ induces an algebraic isomorphism $U(X) \xrightarrow{\sim} U(A_{d,n})$. Now the claim follows from Proposition 11 and Lemma 7. □

Let Z be an irreducible affine SL_n-variety of dimension $n \geq 2$ and $\psi : U(Z) \xrightarrow{\sim} U(A_{d,n})$ be an algebraic isomorphism. Let T be an $(n-1)$-dimensional algebraic subtorus of $U(Z)$. Then, by Lemma 11, we can assume that $\psi(T) = T_d$.

Lemma 13. *Let $\psi : U(Z) \xrightarrow{\sim} U(A_{d,n})$ be as above. Then root subgroups U and $\psi(U)$ have the same weights with respect to T and T_d respectively.*

Proof. Let U be a root subgroup of $U(Z)$ with respect to T and $\mathrm{Lie}\, U = \mathbb{C}\nu$, where ν is a generator. Then $\psi(U)$ is the root subgroup of $U(A_{d,n})$ with respect to T_d. The algebraic isomorphism ψ induces an isomorphism $d\psi_e^u : \mathrm{Lie}\, U \to \mathrm{Lie}\, \psi(U)$. Note that action of T on U induces the action of T on $\mathrm{Lie}\, U$. Then $d\psi_e^u(t \circ \nu \circ t^{-1}) = d\psi_e^u(\chi(t)\nu) = \chi(\psi(t))\psi(\nu)$, where $\chi : T \to \mathbb{C}^*$ is a character. □

Theorem 5. *Let $X = A_{d,n}$, SL_2/T or $\mathrm{SL}_2/N(T)$ and Y be an irreducible affine variety. Let also there is a bijective algebraic homomorphism $U(Y) \to U(X)$. Then*

(a) if $X \cong A_{2,2}$, then $Y \cong \mathrm{SL}_2/T$ or $Y \cong A_{2,2}^s$ for some $s \in \mathbb{N}$,

(b) if $X \cong \mathrm{SL}_2/T$, then $Y \cong \mathrm{SL}_2/T$ or $Y \cong A_{2,2}^s$ for some $s \in \mathbb{N}$,

(c) if $X \cong A_{4,2}$, then $Y \cong \mathrm{SL}_2/N(T)$ or $Y \cong A_{4,2}^s$ for some $s \in \mathbb{N}$,

(d) if $n = 2$ and $X \cong \mathrm{SL}_2/N(T)$, then $Y \cong \mathrm{SL}_2/N(T)$ or $Y \cong A_{4,2}^s$ for some $s \in \mathbb{N}$,

(e) otherwise, $Y \cong A_{d,n}^s$ for some $s \geq 1$.

Proof. Let $\psi : U(X) \to U(Y)$ be an algebraic isomorphism. Proposition 11 implies that $\dim Y \leq \dim X$. Since SL_n acts regularly and non-trivially on X, SL_n also acts non-trivially and regularly on Y.

First, let X be isomorphic to $A_{d,n}$. Then by Lemma 5 and by Proposition 1, normalization of Y, which we denote by $\tilde{Y}$, is isomorphic to SL_2/T, $\mathrm{SL}_2/N(T)$ or $A_{l,n}$ for some $l \geq 1$.

First, assume that $\tilde{Y} \cong A_{l,n}$. Hence, Proposition 5 implies that $\mathcal{O}(Y) = \mathbb{C} \oplus \sum_{i=1}^{r} \sum_{k=k_i}^{\infty} \mathbb{C}[x_1, ..., x_n]_{kl_i}$ for some $r, k_i, l_i \in \mathbb{N}$, $i \in \{1, ..., l\}$. Let $\eta : A_{l,n} \to Y$ be the normalization morphism which by Lemma 4 induces the algebraic homomorphism $\tilde{\eta} : U(Y) \hookrightarrow U(A_{l,n})$. Note that $\mathrm{SL}_n/\mu_{(n,d)}$ acts faithfully on X. Then $\mathrm{SL}_n/\mu_{(n,d)}$ also acts faithfully on Y and therefore on $A_{l,n}$. This implies that $(n,d) = (n,l)$. By Lemma 11, we can assume without loss of generality that $\psi^{-1}(\tilde{\eta}^{-1}(T_l)) = T_d$.

It is clear that for any $s_i \geq k_i$, the group $U = \{(x_1 + cx_2^{s_i d_i + 1}, x_2, ..., x_n) | c \in \mathbb{C}\} \subset \mathrm{Aut}^{\mu_l}(\mathbb{A}^n)$ induces a root subgroup $\bar{U}$ of $U(Y)$ with respect to $\tilde{\eta}^{-1}(T_l)$, and then U acts on $\mathcal{O}(Y)$. Since $(n,d) = (n,l)$, $\phi_d|_{T_n'}$ and $\phi_l|_{T_n'}$ have the same kernels, and because $\bar{U}$ and $\psi^{-1}(\bar{U})$ have the same weights with respect to $\tilde{\eta}^{-1}(T_l)$ and T_d respectively, by Lemma 12, U should also induce a $\mathbb{C}^+$-action on $A_{d,n}$. Hence, U acts on $\mathcal{O}(A_{d,n})$ and then $d + s_i d_i \in \mathbb{N}d$. Since s_i is any natural number greater than or equal to k_i, $d|d_i$ for each i. Therefore, $\mathbb{N}d_1 + ... + \mathbb{N}d_k \subset \mathbb{N}d$.

Analogously as above, for any $k \geq 1$, subgroup $U' = \{(x_1 + cx_2^{kd+1}, x_2, ..., x_n)|c \in \mathbb{C}\} \subset \mathrm{Aut}^{\mu_d}(\mathbb{A}^n)$ induces a root subgroup of $U(A_{d,n})$ with respect T_d. Then U' acts on $\mathcal{O}(Y)$, which implies that $d_i k_i + kd \in (\mathbb{N}_{\geq k_1} d_1 + ... + \mathbb{N}_{\geq k_l} d_l)$ for any i, where $\mathbb{N}_{\geq k} := \{m \in \mathbb{N} | m \geq k\}$. This shows that $\mathbb{N}_{\geq k_1} d_1 + ... + \mathbb{N}_{\geq k_l} d_l = \mathbb{N}_{\geq \min_i \{k_i d_i | i=1,...,l\}} d$.

Now assume that $\tilde{Y}$ is isomorphic to SL_2 / T or to $\mathrm{SL}_2 / N(T)$, then by Proposition 7, $Y = \tilde{Y}$. Then (e) follows from Proposition 10.

Let now $X \cong \mathrm{SL}_2 / T$. Then by Lemma 5, $\tilde{Y}$ can only be isomorphic to SL_2 / T, $\mathrm{SL}_2 / N(T)$ or $A_{2,2}$. By Proposition 10, $\tilde{Y}$ is isomorphic to SL_2 / T or to $A_{2,2}$. If $\tilde{Y} \cong \mathrm{SL}_2 / T$, from Proposition 7, it follows that $Y = \tilde{Y}$. Hence, (b) follows from the first part of the proof. Analogously follows (d). □

Proof of Theorem 2. The isomorphism $\mathrm{Aut}(X) \xrightarrow{\sim} \mathrm{Aut}(A_{d,n})$ induces an algebraic isomorphism $U(X) \to U(A_{d,n})$. Note that X admits a torus action of dimension n. From Theorem 5 it follows that X can only be isomorphic to $A_{d,n}^s$. Since normalization of $A_{d,n}^s$ is equal to $A_{d,n}$, it follows from [FK17] that there is a closed embedding $\mathrm{Aut}(A_{d,n}^s) \hookrightarrow \mathrm{Aut}(A_{d,n})$ of ind-groups and the proof follows from Lemma 4. □

Proof of Theorem 3. Isomorphism $\mathrm{Aut}(X) \xrightarrow{\sim} \mathrm{Aut}(\mathrm{SL}_2 / T)$ induces an algebraic isomorphism $U(X) \to U(\mathrm{SL}_2 / T)$. Then the claim follows from Theorem 5 and Lemma 7. □

Proof of Theorem 4. Follows from Theorem 5. □

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GROUPS OF AUTOMORPHISMS OF DANIELEWSKI SURFACES

MATTHIAS LEUENBERGER AND ANDRIY REGETA

Abstract. We show that subgroups $U(\mathrm{SL}_2/T) \subset \mathrm{Aut}(\mathrm{SL}_2/T)$ and respectively $U(\mathbb{A}^2/\mu_2) \subset \mathrm{Aut}(\mathbb{A}^2/\mu_2)$ generated by $\mathbb{C}^+$-actions are not isomorphic as ind-groups although $U(\mathrm{SL}_2/T)$ and $U(\mathbb{A}^2/\mu_2)$ are algebraically isomorphic i.e. there is an isomorphism $\phi : U(\mathrm{SL}_2/T) \to U(\mathbb{A}^2/\mu_2)$ of abstract groups and the restriction of ϕ to any one-dimensional connected unipotent subgroup is an isomorphism of algebraic groups. We also prove that the Lie subalgebra of the Lie algebra of vector fields $\mathrm{Vec}(D_p)$ generated by locally nilpotent vector fields on D_p is simple.

1. Introduction and Main Results

Our base field is the field of complex numbers $\mathbb{C}$. For an affine variety X the automorphism group $\mathrm{Aut}(X)$ has the structure of an ind-group. We will shortly recall the basic definitions in the following section 2. The classical and most studied example is $\mathrm{Aut}(\mathbb{A}^n)$, the group of automorphism of the affine n-space $\mathbb{A}^n$. Other examples which have gotten a lot of attention in recent years are Danielewski surfaces $D_p = \{(x, y, z) \in \mathbb{A}^3 | xy = p(z)\}$, where $\deg p \geq 2$ and p has no multiple roots. Note that $\mathrm{SL}_2/T \cong D_p = V(xz - y^2 + y) \subset \mathbb{A}^3$ (see [DP09]).

Let X be an affine variety. By $U(X)$ we define the subgroup of $\mathrm{Aut}(X)$ generated by $\mathbb{C}^+$-actions (see [AFK13] for details). Let us denote by μ_2 the cyclic group of order 2, which acts on $\mathbb{A}^2$ in the following way: $\xi \cdot (x, y) = (\xi x, \xi y)$, where $\xi \in \mu_2$. In [Reg17, Proposition 10] it is shown that there is an abstract isomorphism $\phi : U(\mathrm{SL}_2/T) \to U(\mathbb{A}^2/\mu_2)$ such that the restriction of ϕ to any algebraic subgroup $U \cong \mathbb{C}^+$ is an isomorphism of algebraic groups. Note that $U(\mathbb{A}^2/\mu_2)$ is a closed subgroup of $\mathrm{Aut}(\mathbb{A}^2/\mu_2)$ (see [Reg17, Proposition 10]) and $U(\mathrm{SL}_2/T) = \mathrm{Aut}^0(\mathrm{SL}_2/T)$ is a closed subgroup of $\mathrm{Aut}(\mathrm{SL}_2/T)$ (see Proposition 4). Hence, $U(\mathrm{SL}_2/T)$ and $U(\mathbb{A}^2/\mu_2)$ are ind-groups.

Theorem 1. *The ind-groups $U(\mathrm{SL}_2/T)$ and $U(\mathbb{A}^2/\mu_2)$ are not isomorphic.*

In order to prove the above result we show that Lie subalgebras $\mathrm{Lie}\, U(\mathrm{SL}_2/T)$ and $\mathrm{Lie}\, U(\mathbb{A}^2/\mu_2)$ are not isomorphic.

Let $\mathrm{Vec}(D_p)$ be the Lie algebra of vector fields on D_p. Consider the Lie subalgebra $\mathrm{Lie}^{\mathrm{alg}}\, U(D_p) \subset \mathrm{Vec}(D_p)$ generated by all locally-nilpotent vector fields on D_p. We prove that such a Lie algebra is simple.

Theorem 2. *Let D_p be a Danielewski surface, where $\deg p \geq 2$. Then $\mathrm{Lie}^{alg}\, U(D_p)$ is a simple Lie algebra.*

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2. Preliminaries

The notion of an ind-group goes back to Shafarevich who called these objects infinite dimensional groups, see [Sh66], [Sh81]. We refer to [Kum02] and [Kr15] for basic notations in this context.

Definition 1. *By an ind-variety over $\mathbb{C}$ we mean a set V together with an ascending filtration $V_0 \subset V_1 \subset V_2 \subset ... \subset V$ such that the following holds:*
(1) $V = \bigcup_{k\in\mathbb{N}} V_k$;
(2) each V_k has the structure of an algebraic variety;
(3) for all $k \in \mathbb{N}$ the subset $V_k \subset V_{k+1}$ is closed in the Zariski-topology.

A morphism between ind-varieties $V = \bigcup_k V_k$ and $W = \bigcup_m W_m$ is a map $\phi : V \to W$ such that for every k there is an $m \in \mathbb{N}$ such that $\phi(V_k) \subset W_m$ and that the induced map $V_k \to W_m$ is a morphism of varieties. Isomorphisms of ind-varieties are defined in the usual way.

Two filtrations $V = \bigcup_{k\in N} V_k$ and $V = \bigcup_{k\in N} V'_k$ are called equivalent if for any k there is an m such that $V_k \subset V'_m$ is a closed subvariety as well as $V'_k \subset V_m$. Equivalently, the identity map $\mathrm{id} : V = \bigcup_{k\in N} V_k \to V' = \bigcup_{k\in N} V'_k$ is an isomorphism of ind-varieties.

An ind-variety V has a natural topology where $S \subset V$ is open, resp. closed, if $S_k := S \cap V_k \subset V_k$ is open, resp. closed, for all k. Obviously, a locally closed subset $S \subset V$ has a natural structure of an ind-variety. It is called an ind-subvariety. An ind-variety V is called affine if all V_k are affine. Throughout this paper we consider only affine ind-varieties and for simplicity we call them just ind-varieties.

For any ind-variety $V = \bigcup_{k\in\mathbb{N}} V_k$ we can define the tangent space in $x \in V$ in the obvious way. We have $x \in V_k$ for $k \geq k_0$, and $T_x V_k \subset T_x V_{k+1}$ for $k \geq k_0$, and then define

$$T_x V := \lim_{k\geq k_0} T_x V_k,$$

which is a vector space of countable dimension. A morphism $\phi : V \to W$ induces linear maps $d\phi_x : T_x V \to T_{\phi(x)} W$ for every $x \in X$. Clearly, for a k-vector space V of countable dimension and a for any $v \in V$ we have $T_v V = V$ in a canonical way.

The product of two ind-varieties is defined in the obvious way. This allows to give the following definition.

Definition 2. *An ind-variety G is said to be an ind-group if the underlying set G is a group such that the map $G \times G \to G$, taking $(g, h) \mapsto gh^{-1}$, is a morphism.*

Note that any closed subgroup H of G, i.e. H is a subgroup of G and is a closed subset, is again an ind-group under the closed ind-subvariety structure on G. It is clear that a closed subgroup H of an ind-group G is an algebraic group if and only if H is an algebraic subset of G.

If G is an affine ind-group, then $T_e G$ has a natural structure of a Lie algebra which will be denoted by $\mathrm{Lie}\, G$. The structure is obtained by showing that every $A \in T_e G$ defines a unique left-invariant vector field δ_A on G, see [Kum02, Proposition 4.2.2, p. 114].

Definition 3. *An ind-group $G = \bigcup_k G_k$ is called* discrete *if G_k is finite for all k. Clearly, G is discrete if and only if* $\mathrm{Lie}\, G$ *is trivial.*

The next result can be found in [St13] (see also [FK17]).

Proposition 1. *Let X be an affine variety. Then* $\mathrm{Aut}(X)$ *has a natural structure of an affine ind-group.*

Since $\mathrm{Aut}(X)$ has a structure of an ind-group for any affine variety X, we can define a Lie algebra of $\mathrm{Aut}(X)$. It is not difficult to see that $\mathrm{Lie}\,\mathrm{Aut}(X)$ can be injectively embedded into the Lie algebra $\mathrm{Vec}(X)$ of vector fields on X: $\psi : \mathrm{Lie}\,\mathrm{Aut}(X) \hookrightarrow \mathrm{Vec}(X)$. In the future, we will always identify $\mathrm{Lie}\,\mathrm{Aut}(X)$ with its image in $\mathrm{Vec}(X)$. Note that $\mathrm{Lie}\,\mathrm{Aut}(X)$ contains all locally nilpotent vector fields because each such vector field ν corresponds to a unipotent subgroup $U \subset \mathrm{Aut}(X)$, $U \cong \mathbb{C}^+$ and $\nu \in \mathrm{Lie}\,U \subset \mathrm{Lie}\,\mathrm{Aut}(X)$. If $U(X) \subset \mathrm{Aut}(X)$ is a closed subgroup, then similarly, $\mathrm{Lie}\,U(X)$ contains all locally nilpotent vector fields.

The next result which we will use in the future was pointed out to us by Hanspeter Kraft.

Proposition 2. *Let $\phi : G \to H$ be a homomorphism of ind-groups. Then ϕ induces a homomorphism $d\phi_e : \mathrm{Lie}\,G \to \mathrm{Lie}\,H$ of Lie algebras.*

3. Automorphisms of Danielewski surface

Let $p \in \mathbb{C}[t]$ be a polynomial of degree $d \geq 2$ with simple roots. Define the Danielewski-surface $D_p \subset \mathbb{A}^3$ to be the zero set of the irreducible polynomial $xy - p(z)$:

$$D_p = \{(x, y, z) \in \mathbb{A}^3 | xy - p(z)\} \subset \mathbb{A}^3.$$

The following is easy ($\dot{\mathbb{C}} := \mathbb{C} \setminus \{0\}$):

(a) D_p is smooth,

(b) the two projections $\pi_x : D_p \to \mathbb{C}$, $(x, y, z) \mapsto x$ and $\pi_y : D_p \to \mathbb{C}$, $(x, y, z) \mapsto y$ are both smooth,

(c) $(D_p)_x = \pi_x^{-1}(\dot{\mathbb{C}}) \xrightarrow{\sim} \dot{\mathbb{C}} \times \mathbb{C}$, $(x, y, z) \mapsto (x, z)$ and similarly for π_y,

(d) $\pi_x^{-1}(0)$ is the disjoint union of d copies of the affine line $\mathbb{C}$.

For the rest of this section we assume $\deg p > 2$ unless stated otherwise. For every nonzero $f \in \mathbb{C}[t]$ there is a $\mathbb{C}^+$-action α_f on $\dot{\mathbb{C}} \times \mathbb{C}$ given by $\alpha_f(s)(x, z) := (x, z + f(x))$, i.e. by translation with $f(x)$ in the fiber of $x \in \dot{\mathbb{C}}$. It is easy to see that this action extends to an action on D_p if and only if $f(0) = 0$. We denote the corresponding actions on D_p by $\alpha_{x,f}$, respectively $\alpha_{y,f}$. The explicit form is

$$\alpha_{x,f}(s)(x, y, z) = (x, p(z + sf(x)), z + sf(x))$$

and similarly for $\alpha_{y,f}$. The projection $\pi_x : D_p \to \mathbb{C}$ is the quotient for all these actions, and the action on $\pi^{-1}(0)$ is trivial. Note that the corresponding vector fields are given by

$$\nu_{x,f} := p'(z)\frac{f(x)}{x}\frac{\partial}{\partial y} + f(x)\frac{\partial}{\partial z} \text{ and } \nu_{y,f} := p'(z)\frac{f(y)}{y}\frac{\partial}{\partial x} + f(y)\frac{\partial}{\partial z}.$$

Lemma 1. *The map* $\alpha_x : (t\mathbb{C}[t])^+ \to \mathrm{Aut}(D_p)$, $f \mapsto \alpha_{x,f}(1)$*, is a closed immersion of ind-groups.*

Proof. The map is obviously a homomorphism of ind-groups. If we denote by $\rho : \mathrm{Aut}(D_p) \to \mathcal{O}(D_p)$ the map $\phi \mapsto \phi^*(z)$, then this is a morphism of ind-varieties, and the composition $\rho \circ \alpha_x$ maps $f \in t\mathbb{C}[t]$ to $f(x) \in \mathbb{C}[x] \subset \mathcal{O}(D_p)$, hence is a closed immersion. $\square$

Denote by $U_x, U_y \subset \operatorname{Aut}(D_p)$ the image of α_x and α_y. Note that there is also a faithful $\mathbb{C}^*$-action on D_p given by $t(x,y,z) := (tx, t^{-1}y, z)$ which normalizes U_x and U_y. Denote by $T \subset \operatorname{Aut}(D_p)$ the image of $\mathbb{C}^*$. The following result is due to Makar-Limanov.

Proposition 3. *The group* $\operatorname{Aut}(D_p)$ *is generated by* U_x, U_y, T *and a finite subgroup* F *which normalizes* $\langle U_x, U_y, T\rangle$.

Proposition 4. $\operatorname{Aut}(\mathrm{SL}_2/T) = U(\mathrm{SL}_2/T) \rtimes \mu_2$, *where* μ_2 *denotes a cyclic group of order* 2. *In particular,* $\operatorname{Aut}(\mathrm{SL}_2/T)^0 = U(\mathrm{SL}_2/T)$ *is an ind-group.*

Proof. By [DP09], $\mathrm{SL}_2/T \cong D_p$, where $\deg p = 2$. Note that for any two polynomials $p, q \in \mathbb{C}[z]$ of degree 2 without multiple roots, we have $D_p \cong D_q$. It follows from [Lam05, Theorem 6] that $\operatorname{Aut}(D_p)$ is generated by $\mathbb{C}^+$-actions and cyclic subgroup μ_2 of order 2 which permute roots $\{a, b\}$ of p, i.e $\operatorname{Aut}(D_p) = \langle U(D_p), \mu_2\rangle$. Because $U(D_p)$ is normal subgroup of $\operatorname{Aut}(D_p)$, we have $\operatorname{Aut}(D_p) = U(D_p) \rtimes \mu_2$. Then $U(D_p) = \{\phi \in \operatorname{Aut}(D_p) | \phi(a) = a, \phi(b) = b\}$ is the closed subgroup of $\operatorname{Aut}(D_p)$. $\square$

We denote by $\operatorname{Lie}^{\mathrm{alg}} U(\mathbb{A}^2/\mu_2)$ the Lie subalgebra of $\operatorname{Vec}(\mathbb{A}^2/\mu_2)$ generated by all locally nilpotent vector fields on $\mathbb{A}^2/\mu_2$.

Proof of Theorem 1. Assume there is an isomorphism $\phi : U(\mathrm{SL}_2/T) \to U(\mathbb{A}^2/\mu_2)$ of ind-groups. By Proposition 2 it induces an isomorphism $d\phi_e : \operatorname{Lie} U(\mathrm{SL}_2/T) \to \operatorname{Lie} U(\mathbb{A}^2/\mu_2)$ of Lie algebras, and because ϕ maps each closed unipotent subgroup $U \cong \mathbb{C}^+$ to $\phi(U) \cong \mathbb{C}^+$, $d\phi_e$ induces an isomorphism of Lie algebras $\operatorname{Lie}^{\mathrm{alg}} U(\mathrm{SL}_2/T)$ and $\operatorname{Lie}^{\mathrm{alg}} U(\mathbb{A}^2/\mu_2)$.

By Theorem 2, $\operatorname{Lie}^{\mathrm{alg}} U(\mathrm{SL}_2/T)$ is simple. On the other hand, we claim that $\operatorname{Lie}^{\mathrm{alg}} U(\mathbb{A}^2/\mu_2)$ is not simple. Indeed, since $\mathbb{A}^2/\mu_2$ has an isolated singular point s, each vector field, which comes from an algebraic group action, vanishes at this singular point. In particular, each locally nilpotent vector field vanishes at isolated singular point. Because $\operatorname{Lie}^{\mathrm{alg}} U(\mathbb{A}^2/\mu_2)$ is generated by locally nilpotent vector fields, each $\nu \in \operatorname{Lie}^{\mathrm{alg}} U(\mathbb{A}^2/\mu_2)$ vanishes at isolated singular point of $\mathbb{A}^2/\mu_2$. Let $I \subset \operatorname{Lie}^{\mathrm{alg}} U(\mathbb{A}^2/\mu_2)$ be a Lie subalgebra generated by those vector fields which vanish at isolated singular point with multiplicity $k > 1$. It is clear that $I \neq \operatorname{Lie}^{\mathrm{alg}} U(\mathbb{A}^2/\mu_2)$ because $x\frac{\partial}{\partial y} \in (\operatorname{Lie}^{\mathrm{alg}} U(\mathbb{A}^2/\mu_2) \setminus I)$. Moreover, it is clear that $[\nu, \mu] \in I$ for any $\nu \in I$ and $\mu \in \operatorname{Lie}^{\mathrm{alg}} U(\mathbb{A}^2/\mu_2)$ which shows that I is an ideal. The claim follows. $\square$

4. Module of differentials and vector fields

Since D_p is smooth, the differentials $\Omega(D_p)$ and the vector fields $\operatorname{Vec}(D_p) \xrightarrow{\sim} \operatorname{Hom}(\Omega(D_p), \mathcal{O}(D_p))$ are locally free $\mathcal{O}(D_p)$-modules, and then, projective. More precisely, we have the following description.

Proposition 5. (*a*) *The module* $\Omega(D_p)$ *of differentials is projective of rank* 2 *and is generated by* dx, dy, dz, *with the unique relation* $ydx + xdy - p'(z)dz = 0$.

(*b*) *The module* $\Omega^2(D_p) := \bigwedge^2 \Omega(D_p)$ *is free of rank one and is generated by*

$$\omega := \frac{1}{x} dx \wedge dz = \frac{1}{y} dy \wedge dz = \frac{1}{p'(z)} dx \wedge dy.$$

Proof. (a) From above it is clear that $\Omega(D_p)$ is the projective module of rank $2 = \dim(D_p)$. It is easy to see that $\Omega(D_p) = (\mathcal{O}(D_p)dx \oplus \mathcal{O}(D_p)dy \oplus \mathcal{O}(D_p)dz)/(ydx +$

$xdy - p'(z)dz)$, where $ydx + xdy - p'(z)dz = d(xy - p(z))$. In fact, the surface D_p is covered by the special open sets $D_x, D_y, D_{p'(z)}$ and $\Omega(D_p)$ is free module of rank two over these open sets, generated by (dx, dz), by (dy, dz), and by (dx, dy), respectively.

(b) The three expressions are well-defined in the special open sets $D_x, D_y, D_{p'(z)}$, respectively, and the relation $ydx + xdy - p'(z)dz = 0$ implies that they coincide on the intersections. Thus ω is a nowhere vanishing section of $\Omega^2(D_p)$ and therefore, $\Omega^2(D_p)$ is free of rank 1 (see also [KK10, Section 3] for details). $\square$

Remark 1. *In fact, for any normal hypersurface $X \subset \mathbb{A}^n$, $\Omega^{n-1}(X) := \bigwedge^{n-1} \Omega(X)$ is free of rank one.*

Remark 2. *Note that there is no $\delta \in \mathrm{Vec}(D_p)$ such that $\delta : \mathcal{O}(D_p) \to \mathcal{O}(D_p)$ is surjective because $\Omega(D_p)$ is not free. Note also that ω is unique up to a constant because $\mathcal{O}(D_p)^* = \mathbb{C}^*$.*

It is well-known that every vector field δ on $D_p \subset \mathbb{A}^3$ extends to a vector field $\tilde{\delta}$ on $\mathbb{C}^3$. It follows that δ can be written in the form

$$\delta = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z},$$

where $a, b, c \in \mathcal{O}(D_p)$ such that $ay + bx - cp'(z) = 0$ in $\mathcal{O}(D_p)$. In fact, considering δ as a $\mathcal{O}(D_p)$-linear map $\Omega(D_p) \to \mathcal{O}(D_p)$, we have $a = \delta(dx)$, $b = \delta(dy)$ and $c = \delta(dz)$. This presentation of δ is unique.

Remark 3. *In fact, the vector fields $\mathrm{Vec}(D_p)$ form a module over $\mathcal{O}(D_p)$ of rank 2, generated by*

$$\nu_z := x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, \;\; \nu_x := p'(z)\frac{\partial}{\partial y} + x\frac{\partial}{\partial z}, \;\; \nu_y := p'(z)\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}$$

with the unique relation $x\nu_y - y\nu_x = p'(z)\nu_0$.

The next result is clear.

Proposition 6. *The sequence*

$$0 \to \mathbb{C} \to \mathcal{O}(D_p) \xrightarrow{d} d\Omega(D_p) \xrightarrow{d} d\Omega^2(D_p) \to 0$$

is exact.

5. Volume form and divergence.

For any $\theta \in \mathrm{Vec}(D_p)$ we have the contraction

$$i_\theta : \Omega^{k+1} \to \Omega^k, \;\; i_\theta(\eta)(\theta_1, ..., \theta_k) := \eta(\theta, \theta_1, ..., \theta_k).$$

In particular, for $\eta \in \Omega(D_p)$, we have $i_\theta(\eta) = \eta(\theta) \in \mathcal{O}(D_p)$, and so $i_\theta(df) = \theta_f$.

The vector field $\theta \in \mathrm{Vec}(D_p)$ acts on the differential forms $\Omega(D_p)$ by the Lie derivative $L_\theta := d \circ i_\theta + i_\theta \circ d$, extending the action on $\mathcal{O}(D_p)$. One finds (see for details [KK10, Section 3])

$L_\theta(f) = \theta f$, $L_\theta(df) = d(\theta f)$ and $L_\theta(h \cdot \mu) = \theta h \cdot \mu + h \cdot L_\theta \mu$ for $f, h \in \mathcal{O}(D_p), \mu \in \Omega(D_p)$.

Using the volume form ω (see Proposition 5), this allows to define the divergence $\mathrm{div}(\theta)$ of a vector field θ:

$$L_\theta \omega = d(i_\theta \omega) = \mathrm{div}(\theta) \cdot \omega.$$

Lemma 2. *(Hanspeter Kraft). Let* $\theta = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z} \in \mathrm{Vec}(D_p)$*. Then* $\mathrm{div}(\delta) = a_x + b_y + c_z$*.*

Proof. We have $i_\theta\omega = \frac{1}{x}(\theta(x)dz - \theta(z))dx = \frac{1}{x}(adz - cdx)$, hence

$$\begin{aligned}\mathrm{div}(\theta)\cdot\omega = d(i_\theta\omega) =& d(\frac{1}{x}(adz - cdx))\\ =& \frac{1}{x^2}((xda - adx)\wedge dz - (xdc \wedge dx).\end{aligned}$$

Now we use the following equalities: $da \wedge dz = a_x \cdot dx \wedge dz + a_y \cdot dy \wedge dz$, $dx \wedge dc = c_y \cdot dx \wedge dy + c_x \cdot dx \wedge dz$, $dy \wedge dz = y \cdot \omega$, and $dx \wedge dy = p'(z)\cdot\omega$ (see above) to get

$$\mathrm{div}(\theta) = -\frac{a}{x} + a_x - \frac{y}{x}a_y + \frac{p'(z)}{x}c_y + c_z.$$

Since $ya + xb - p'(z)c = 0$ we have $a + ya_y + xb_y - p'(z)c_y = 0$, hence

$$-\frac{a}{x} - \frac{y}{x}a_y + \frac{p'(z)}{x}c_y = b_y,$$

and the claim follows. □

There is another important formula which relates the Lie structure of $\mathrm{Vec}(D_p)$ with the Lie derivative (see also [KL13, Lemma 3.2]).

Lemma 3. *For* $\theta_1, \theta_2 \in \mathrm{Vec}(D_p)$ *and* $\mu \in \Omega(D_p)$ *we have*

$$i_{[\theta_1,\theta_2]}\mu = L_{\theta_1}(i_{\theta_2}\mu) - i_{\theta_2}(L_{\theta_1}\mu).$$

6. Duality.

The volume form $\omega \in \Omega^2(D_p)$ induces the usual duality between vector fields and differential forms: the $\mathcal{O}(D_p)$-isomorphism $\mathrm{Vec}(D_p) \xrightarrow{\sim} \Omega(D_p)$ is given by $\theta \mapsto i_\theta\omega$. In particular, for every $f \in \mathcal{O}(D_p)$ we get a vector field $\nu_f \in \mathrm{Vec}(D_p)$ defined by $i_f\omega = df$, i.e. $df \wedge \eta = \nu_f(\eta)\cdot\omega$.

Denote by $\mathrm{Vec}^0(D_p) \subset \mathrm{Vec}(D_p)$ the subspace of volume preserving vector fields, i.e. $\mathrm{Vec}^0(D_p) := \{\theta \in \mathrm{Vec}(D_p) |\, \mathrm{div}\,\theta = 0\}$.

Proposition 7. *The map* $f \mapsto \nu_f$ *induces a* $\mathbb{C}$*-linear isomorphism*

$$\mathcal{O}(D)/\mathbb{C} \xrightarrow{\sim} \mathrm{Vec}^0(D_p).$$

Proof. Since $d(i_\theta) = \mathrm{div}(\theta)\cdot\omega$, we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{C} & \xrightarrow{d} & \mathcal{O}(D_p) & \xrightarrow{d} & \Omega(D_p) & \xrightarrow{d} & \Omega^2(D_p) & \longrightarrow & 0 \\ & & & & \big\uparrow = & & \simeq\big\uparrow \theta\mapsto i_\theta\omega & & \simeq\big\uparrow h\mapsto h\cdot\omega & & \\ & & & & \mathcal{O}(D_p) & \xrightarrow{\nu} & \mathrm{Vec}(D_p) & \xrightarrow{\mathrm{div}} & \mathcal{O}(D_p) & & \end{array}$$

Now the claim follows because the first row is exact (see Proposition 6). □

The following result can be found in [KL13, Theorem 3.26].

Proposition 8. *Any vector field $\nu \in \mathrm{Vec}^0(D_p)$ on the Danielewski surface D_p is a Lie combination of locally nilpotent vector fields if and only if its corresponding function with $i_\nu \omega = df$ is of the form (modulo constant)*

$$f(x,y,z) = \sum_{i=1,j=0}^{k} a_{ij} x^i z^j + \sum_{i=1,j=0}^{l} b_{ij} y^i z^j + (pq)'(z) \tag{1}$$

for a polynomial $q \in \mathbb{C}[z]$. Any $f \in \mathcal{O}(D_p)$ bijectively corresponds to some $\nu_f \in \mathrm{Vec}^0(D_p) := \mathrm{Lie}^{alg}\, U(D_p) \oplus \bigoplus_{i=0}^{\deg p-2} \mathbb{C} z^i (x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y})$.

The corresponding functions are given as follows (see [KL13, Lemma 3.1]):

$$f_{x^i\nu_x} = -\frac{x^{i+1}}{i+1}, \quad f_{y^i\nu_y} = \frac{y^{i+1}}{i+1}, \quad f_{\nu_z^q} = (p(z)q(z))'. \tag{2}$$

We also recall the useful relation that describes the corresponding function of a Lie bracket of two vector fields $\nu, \mu \in \mathrm{Lie}^{\mathrm{alg}}\, U(D_p)$ (see [KL13, Lemma 3.2]):

$$f_{[\nu,\mu]} = \nu(f_\mu), \tag{3}$$

where $\nu(f_\mu)$ is ν applied as a derivation to the function f_μ. The function $f_{[\nu,\mu]}$ may also be calculated by the following formula (see [KL13, formula after Lemma 3.2]):

$$\begin{aligned} f_{[\nu,\mu]} = \{f_\nu, f_\mu\} := p'(z)\big((f_\nu)_y (f_\mu)_x - (f_\nu)_x (f_\mu)_y\big) + \\ x\big((f_\nu)_z (f_\mu)_x - (f_\nu)_x (f_\mu)_z\big) - y\big((f_\nu)_z (f_\mu)_y - (f_\nu)_y (f_\mu)_z\big), \end{aligned} \tag{4}$$

where the subindex denotes the partial derivative to the respective variable.

Let $I \subset \mathrm{Lie}^{\mathrm{alg}}\, U(D_p)$ be a non-trivial ideal and let $\tilde{I}$ be the set of functions corresponding to this ideal by the correspondence in (1). Since I is an ideal, we have, using (3), that

$$\left\{\nu \in I \left(\Leftrightarrow f_\nu \in \tilde{I}\right) \text{ and } \mu \in \mathrm{Lie}^{\mathrm{alg}}\, U(D_p)\right\} \implies \nu(f_\mu),\ \mu(f_\nu) \in \tilde{I}. \tag{5}$$

The algebraic vector fields $\nu_{x^i} := p'(z)x^i \frac{\partial}{\partial y} + x^{i+1}\frac{\partial}{\partial z}$, $\nu_{y^i} := p'(z)y^i\frac{\partial}{\partial x} + y^{i+1}\frac{\partial}{\partial z}$ on the Danielewski surface D_p are called shear fields for all $i \in \mathbb{N}_0$, and the vector fields $\nu_z^h := h'(z)(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y})$ are called hyperbolic fields for all $h \in \mathbb{C}[z]$.

Our next goal is to prove Theorem 2. We prove it in several steps and start with the following Lemma.

Lemma 4. *Let f be a regular function on D_p. Then f can be written uniquely as $f(x,y,z) = \sum_{i=l}^{k} a_i(z)x^i$ for some $k, l \in \mathbb{Z}$.*

Proof. Let us take the form of f as in (1) and replace y by $p(z)/x$. This yields to $a_i(z) = b_i(z)p_i(z)$ for $i \in \mathbb{N}$. □

Choose $l, k \in \mathbb{Z}$ such that $a_l, a_k \neq 0$ and denote by $\deg(f) = (l,k)$ the pair of min- and max-degree in x.

Lemma 5. *Let $f \in \mathcal{O}(D_p)$. Then $\nu_x(f)$ and $\nu_y(f)$ are never non-zero constants.*

Proof. Any regular function f of D_p can be written in the form $\sum_{i=l}^{k} a_i(z)x^i$ by Lemma 4. Then $\nu_x(f) = \sum_{i=l}^{k} a_i'(z)x^{i+1}$, in particular, $\nu_x(f)$ is constant only if a_{-1} is linear, which is not the case since a_{-1} is divisible by p. The case of $\nu_y(f)$ is analogous. □

Lemma 6. *Let $\deg f = (l,k)$ and $l, k \geq 1$. Then $\deg \nu_y(f) = (l-1, k-1)$.*

Proof. Let $f = \sum_{i=l}^{k} a_i(z)x^i$. Then $\nu_y(f) = \sum_{i=l}^{k}(ip'(z)a_i(z) + p(z)a_i'(z))x^{i-1}$. If $a_i(z) \neq 0$, then $ip'(z)a_i(z) + p(z)a_i'(z) \neq 0$ and the claim follows. □

Lemma 7. *Let* $\deg f = (l, k)$*, where* $k > l \geq 0$*, then* $\deg \nu_z^1(f) = (\tilde{l}, k)$*, where* $\tilde{l} = l$ *if* $l \geq 1$ *and* $\tilde{l} > l$ *if* $l = 0$*.*

Proof. Let $f = \sum_{i=l}^{k} a_i(z)x^i$. Then the claim follows from the equality $\nu_z^1(f) = \sum_{i=\tilde{l}}^{k} ip''(z)a_i(z)x^i$. □

Proof of Theorem 2. Let I be a nontrivial ideal of $\mathrm{Lie}^{\mathrm{alg}}\, U(D_p)$. Then there is a nonzero $f \in \tilde{I}$, and since ν_x is locally nilpotent, there is $k \in \mathbb{N}$ such that $\nu(\nu^k(f)) = 0$, and then $\nu^k(f) \in \mathbb{C}[x] \setminus \mathbb{C}$. Therefore, there is $g \in \tilde{I}$ such that $g \in \tilde{I}$ and $\deg g = (l, k)$, where $k, l \geq 1$. By applying Lemma 6 and Lemma 7 step by step, we will get that here is $h \in \tilde{I}$ such that $\deg h = (0, 0)$. Therefore, by Lemma 8, $q(z)x^n \in \tilde{I}$ for all $q \in \mathbb{C}[z]$ and $n \geq 1$.

Analogously, interchanging x and y, we get that $q(z)y^i \in \tilde{I}$ for all $q(z) \in \mathbb{C}[z]$ and $i \in \mathbb{N}_0$. Since $q(z)x \in \tilde{I}$ for all $q(z)$, also $\nu_y(q(z)x) = (p(z)q(z)) \in \tilde{I}$ for all $q(z)$, by (4). Thus $\tilde{I}$ contains all functions that correspond to vector fields in $\mathrm{Lie}^{\mathrm{alg}}\, U(D_p)$ or, equivalently, $I = \mathrm{Lie}^{\mathrm{alg}}\, U(D_p)$, which concludes the proof. □

Lemma 8. *Let* $h \in \tilde{I}$*,* $h \in \mathbb{C}[z] \setminus \mathbb{C}$*, then* $q(z)x^n \in \tilde{I}$ *for all* $q \in \mathbb{C}[z]$ *and* $n \geq 1$*.*

Proof. First we claim that there is an $N \in \mathbb{N}$ such that $q(z)x^{n+1} \in \tilde{I}$ for all $q \in \mathbb{C}[z]$ and $n \geq N$. By (5), we get that $\nu_x(h) = h'(z)x \in \tilde{I}$ and $\nu_z^s(h'(z)x) = (p(z)s(z))''h'(z)x \in \tilde{I}$ for all $s \in C[z]$. Let $N = \deg p''h'$ and $n \geq N$. Then applying (5) $N - 1$ times for ν_x we get

$$\nu_x^{N-1}((p(z)s(z))''h'(z)x) = ((ps)''h')^{N-1}(z)x^N \in \tilde{I}.$$

Now apply (5) once more for $x^{n-N}\nu_x$ and get

$$x^{n-N}\nu_x(((ps)''h')^{N-1}(z)x^N) = ((ps)''h')(N)(z)x^{n+1} \in \tilde{I},$$

and thus varying $s(z)$ we get $q(z)x^{n+1} \in \tilde{I}$ for all $q \in \mathbb{C}[z]$.

Hence

$$\nu_y(z^j x^n) = ip'(z)z^j x^{n-1} + jp(z)z^{j-1}x^{n-1} \in \tilde{I}$$

for all $j \in \mathbb{N} \cup \{0\}$. On the other hand, by the assumption $x^i \in \tilde{I}$, and thus by (2), we have $x^{i-1}\nu_x \in \tilde{I}$. Hence, by (4),

$$x^{i-1}\nu_x(yz^j) = p'(z)z^j x^{i-1} + jp(z)z^{j1}x^{i-1} \in \tilde{I}$$

for all $j \in \mathbb{N} \cup \{0\}$. By taking suitable linear combinations of the above expressions we see that $x^{i-1} \cdot (p'(z)) \subset \tilde{I}$ and $x^{i-1} \cdot (p(z)) \subset \tilde{I}$, where $(p'(z))$ and $(p(z))$ denote the ideal in $\mathbb{C}[z]$ generated by $p'(z)$ and the ideal generated by $p(z)$. Since p has no simple roots by assumption, the ideal $(p'(z), p(z))$ generated by both $p'(z)$ and $p(z)$ is equal to $\mathbb{C}[z]$ and thus $x^{i-1} \cdot (p'(z), p(z)) = x^{i-1} \cdot \mathbb{C}[z] \subset \tilde{I}$. □

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Mathematisches Institut, Universität Bern, Sidlerstrasse 5 CH-3012 Bern
E-mail address: `matthias.leuenberger@math.unibe.ch`

Mathematisches Institut, Universität Basel, Spiegelgasse 1, CH-4051 Basel
E-mail address: `andriy.regeta@unibas.ch`