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# Characterization of smooth Danielewski surfaces by their automorphism groups

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## Abstract

In dieser Masterarbeit untersuchen wir sogenannte Danielewski Oberflächen $D_p = \{(x, y, z) \in \mathbb{A}^3 \mid xy = p(z)\} \subseteq \mathbb{A}^3$ wobei $\mathbb{A}^3$ der affine 3-Raum und $p \in \mathbb{C}[z]$ ein Polynom mit einfachen Nullstellen ist. Dabei zeigen wir, falls $X$ eine normale irreduzible affine Varität ist, so dass die Automorphismengruppe $\mathrm{Aut}(X)$ isomorph zu $\mathrm{Aut}(D_p)$ ist, muss $X$ isomorph zu $D_q = \{(x, y, z) \in \mathbb{A}^3 \mid xy = p(z)\} \subseteq \mathbb{A}^3$ sein, wobei $q \in \mathbb{C}[z]$ ein Polynom mit möglicherweise mehrfache Nullstellen ist. Falls $X$ glatt ist und $\mathrm{Aut}(X)$ isomorph als eine Ind-Gruppe ist, so ist $X$ isomorph zu $D_p$ als Varitäten.

## Abstract

In this master thesis we study so-called Danielewski surfaces $D_p = \{(x, y, z) \in \mathbb{A}^3 \mid xy = p(z)\} \subset \mathbb{A}^3$, where $\mathbb{A}^3$ is the affine 3-space and $p \in \mathbb{C}[z]$ is a polynomial with simple roots. We show that if $X$ is a normal irreducible affine variety such that the group of automorphism $\mathrm{Aut}(X)$ is isomorphic to $\mathrm{Aut}(D_p)$, then $X$ is isomorphic to $D_q = \{(x, y, z) \in \mathbb{A}^3 \mid xy = q(z)\} \subset \mathbb{A}^3$, where $q \in \mathbb{C}[z]$ is a polynomial with possibly multiple roots. Additionally, if $X$ is smooth and $\mathrm{Aut}(X)$ is isomorphic to $\mathrm{Aut}(D_p)$ as an ind-group, $X$ is isomorphic to $D_p$ as a variety.

# Contents

# Nomenclature

| | |
|---|---|
| $\mathrm{Aut}(X)$ | automorphism group on $X$ |
| $\mathcal{O}(X)$ | $\mathbb{C}$-algebra of regular functions on $X$ |
| $\mathcal{O}(X)^G$ | set of all $G$-invariants on $\mathcal{O}(X)$ |
| $\mathrm{codim}_X(Z)$ | codimension of $Z \subseteq X$ |
| $\deg p$ | degree of the polynomial $p$ |
| $\det(A)$ | determinant of the matrix $A$ |
| $\dim(X)$ | dimension of the variety $X$ |
| $\mathrm{End}(X)$ | set of endomorphisms on $X$ |
| $\frac{\partial f}{\partial x_i}$ | formal partial derivative of $f$ by $x_i$ |
| $\mathrm{GL}_n$ | general linear group of $n \times n$ matrices |
| $\mathbb{A}^n$ | affine $n$-space |
| $\mathbb{C}$ | complex numbers |
| $\mathbb{C}[x]$ | polynomial ring over the field $\mathbb{C}$ |
| $\mathbb{C}^*$ | multiplicative group of $\mathbb{C} \setminus \{0\}$ |
| $\mathbb{C}^+$ | additive group of $\mathbb{C}$ |
| $\mathbb{N}$ | natural numbers containing 0 |
| $\mathbb{N}_1$ | natural numbers not containing 0 |
| $\mathbb{Z}$ | all integers |
| $\mathrm{Kdim}(R)$ | Krull dimension of $R$ |
| $\mathrm{Mor}(X, Y)$ | set of morphisms from $X$ to $Y$ |
| $\overline{C}$ | closure of $C$ |
| $\mathscr{I}(Y)$ | vanishing ideal of $Y \subseteq \mathbb{A}^n$ |
| $\sqrt{I}$ | radical ideal of the ideal $I$ |
| $\mathrm{SL}_n$ | special linear group of $n \times n$ matrices |

| | |
|---|---|
| $\mathrm{Stab}_G(x)$ | stabilizer of $x$ under $G$ action |
| $\mathscr{Z}(S)$ | zero set for $S \subseteq \mathbb{C}[x_1, \dots, x_n]$ |
| $D_p$ | Danielewski surface of polynomial $p$ |
| $G * H$ | free product of $G$ and $H$ |
| $G \cdot x$ | orbit of $x$ under $G$ action |
| $G \rtimes H$ | semidirect product of $G$ and $H$ |
| $G \times H$ | direct product of $G$ and $H$ |
| $G^\circ$ | connected component of the neutral element of $G$ |
| $p'(x)$ | formal derivative of the polynomial $p$ |
| $x\mathbb{C}[x]^+$ | additive group of the polynomials $x\mathbb{C}[x]$ |

# 0 Introduction

In 1872, Felix Klein suggested in his famous Erlangen Programm to study geometrical objects through their symmetries. In the spirit of this program it is natural to ask to which extent a geometrical object is determined by its group of symmetries. As an example, a smooth manifold, a symplectic manifold or a contact manifold is determined by its group of symmetries, see [2], [20], [21]. In this thesis we study the following question:

**0.1 Question.** To which extend is an irreducible affine algebraic variety determined by its group of regular automorphisms?

Throughout this thesis we work over the field of complex numbers $\mathbb{C}$ and algebraic varieties are always considered to be affine. For a variety $X$ we denote by $\mathrm{Aut}(X)$ the group of automorphisms of $X$. As the automorphism group of a variety is usually quite small, it almost never determines the variety. However, if $\mathrm{Aut}(X)$ is large, it might do. A good example of an affine variety with a big automorphism group is the so-called toric variety. More precisely, let $T$ be the complex algebraic torus, i.e. $T \cong (\mathbb{C}^*)^n$, where $\mathbb{C}^*$ is the multiplicative group of the base field $\mathbb{C}$. A toric variety $X$ is a normal algebraic variety endowed with a regular and faithful action of $T$ such that $T$ acts on $X$ with an open orbit. The following result is proved in [18].

**0.2 Theorem.** Let $X$ be an affine toric variety different from the algebraic torus, and let $Y$ be a normal affine variety. If $\mathrm{Aut}(X)$ and $\mathrm{Aut}(Y)$ are isomorphic as abstract groups, then $X$ and $Y$ are isomorphic.

A similar result is unknown for normal affine irreducible surfaces with a big automorhism group, i.e., those surfaces that admit actions of non-commuting $\mathbb{C}^+$-actions. By [3, Theorem 3.3] any such surface is a quotient of a so-called Danielewski surface $D_p = \{(x, y, z) \in \mathbb{A}^3 \mid xy = p(z)\} \subset \mathbb{A}^3$ for some polynomial $p \in \mathbb{C}[z]$ by a finite cyclic group. So, to prove a similar result to Theorem 0.2, we need first to characterize a Danielewski surface by its group of automorphisms. Therefore, our guiding question is the following.

**0.3 Question.** Let $X$ be an affine irreducible variety such that $\mathrm{Aut}(X)$ is isomorphic to $\mathrm{Aut}(D_p)$. Are varieties $X$ and $D_p$ isomorphic?

We call a surface $D_p$ *generic* if there is no affine automorphism of the affine line $\mathbb{C}$ that permutes the roots of the polynomial $p$. For two generic surfaces $D_p$ and $D_q$ with $\deg p \geq 3$ and $\deg q \geq 3$ there is an isomorphism $\mathrm{Aut}(D_p) \xrightarrow{\sim} \mathrm{Aut}(D_q)$ of abstract groups.

Indeed, in [12, Theorem and Remark (3) on p. 256] and more precisely in [8, Theorem 2.7] it is shown that for a generic Danielewski surface $D_p$, we have $\mathrm{Aut}(D_p) \simeq (\mathbb{C}[x] * \mathbb{C}[y]) \rtimes (\mathbb{C}^* \rtimes \mathbb{Z}/2\mathbb{Z})$ and the semidirect product structure does not depend on $p(z)$ (see also [9, Remark 7]). On the other hand, by [9, Theorem 3] $\mathrm{Aut}(D_p)$ and $\mathrm{Aut}(D_q)$ are isomorphic as ind-groups, if and only if $D_p$ is isomorphic to $D_q$ as a variety. In this thesis we prove the following results.

**Main Theorem (A).** Let $X$ be an affine irreducible normal variety and $p \in \mathbb{C}[z]$ be a polynomial with simple roots. If $\varphi : \mathrm{Aut}(D_p) \to \mathrm{Aut}(X)$ is an isomorphism of abstract groups and $\deg p \neq 2$, then $X$ is isomorphic to $D_q$ for some polynomial $q \in \mathbb{C}[z]$.

**Main Theorem (B).** Let $X$ be a smooth and irreducible affine variety. If the automorphism group $\mathrm{Aut}(X)$ is isomorphic to $\mathrm{Aut}(D_p)$ as an ind-group for a polynomial $p$ with simple roots, then $X$ is isomorphic to $D_p$.

This thesis is organized as follows. In the first chapter we present basic results about affine algebraic varieties, linear algebraic groups and action of such groups on affine varieties. In the second chapter we introduce the notions of an ind-variety and an ind-group. This notion is crucial for us as the automorphism group of an affine variety has a natural structure of an ind-group (see Theorem 2.6). The last part of the chapter two is devoted to the proof of Main Theorem A and Main Theorem B in the case $\deg p \leq 2$.

In the third chapter we study the structure of the automorphism group $\mathrm{Aut}(D_p)$ of a Danielewski surface $D_p$, where the degree of $p \in \mathbb{C}[z]$ is bigger than two and root subgroups of $\mathrm{Aut}(D_p)$ (see Defnition 3.10). Moreover, we show that root subgroups of $\mathrm{Aut}(D_p)$ are uniquely determined by their weights.

The last chapter is devoted to the proofs of Main Theorem A and Main Theorem B. More precisely, we show that if $X$ is an affine irreducible algebraic variety and $\varphi$: $\mathrm{Aut}(D_p) \to \mathrm{Aut}(X)$ is an abstract group isomorphism, then the image of a certain algebraic subgroup $T \subset \mathrm{Aut}(D_p)$ is an algebraic subgroup of $\mathrm{Aut}(X)$ isomorphic to $\mathbb{C}^*$ (see Lemma 4.2). Moreover, root subgroups of $\mathrm{Aut}(D_p)$ with respect to $T$ are sent to root subgroups of $\mathrm{Aut}(X)$ with respect to $\varphi(T)$. Using [7, Lemma 5.2] this allows us to show that $\dim X = 2$ (Theorem 4.10). We conclude the result of Main Theorem A by applying [11, Theorem 1]. Main Theorem B follows from Main Theorem A and [9, Theorem 3].

# 1 Preliminaries

## 1.1 The affine space and Zariski topology

For $n \in \mathbb{N}_1$ we define $\mathbb{A}^n = \mathbb{C}^n$ which is called the *affine n-space*. Before we define affine varieties, we first need to define the topology on this space. We consider the *zero set* for $S \subseteq \mathbb{C}[x_1, \dots, x_n]$ which is defined as

$$\mathscr{Z}(S) = \{P = (a_1, \dots, a_n) \in \mathbb{A}^n \mid f(P) = 0 \text{ for all } f \in S\}.$$

With zero sets we can define a topology on $\mathbb{A}^n$, where $\mathscr{Z}(S)$ are *closed* subsets for $S \subseteq \mathbb{C}[x_1, \dots, x_n]$. Define an *open* subset as the complement of a closed subset. This topology is called the *Zariski topology*, and we always use it in the context of varieties.

Similar to the zero set we define the *vanishing ideal* of a subset $Y \subseteq \mathbb{A}^n$ in the following way :

$$\mathscr{I}(Y) = \{f \in \mathbb{C}[x_1, \dots, x_n] \mid f(P) = 0 \text{ for all } P \in Y\}.$$

Let $I$ be an ideal of $\mathbb{C}[x_1, \dots, x_n]$. Then the *radical ideal* of $I$ is

$$\sqrt{I} := \{r \in \mathbb{C}[x_1, \dots, x_n] \mid \exists m \in \mathbb{N} : r^m \in I\}.$$

**1.1 Theorem** (Hilbert's Nullstellensatz)**.** If $I \subseteq \mathbb{C}[x_1, \dots, x_n]$ is an ideal, then $\mathscr{I}(\mathscr{Z}(I)) = \sqrt{I}$.

Hilbert's Nullstellensatz shows that $\mathscr{I}$ and $\mathscr{Z}$ are bijections between the closed subsets in $\mathbb{A}^n$ and the radical ideals of $\mathbb{C}[x_1, \dots, x_n]$.

The *closure* of $C$ in a topological space $X$ is denoted by $\overline{C}$ and is defined as the intersection of all closed subsets $A \subseteq X$ with $C \subseteq A$. It can be thought of as the smallest closed subset that contains $C$. The set $C$ is *dense* if $\overline{C} = X$.

A topological space $X$ is *reducible*, if $X = \emptyset$ or if $X$ is the union of two proper closed subsets. The space $X$ is *irreducible*, if it is not reducible. A subset of a topological space is called *reducible*, if it is reducible as a topological space with the induced topology. Note that every non-empty open subset $U$ of an irreducible topological space is dense. This follows, because the closure of $U$ and the complement of $U$ would otherwise form a decomposition of the irreducible space.

**1.2 Example.** For $n \in \mathbb{N}$ the affine $n$-space $\mathbb{A}^n$ with the Zariski topology is irreducible. Let's suppose that it is reducible, say $\mathbb{A}^n = Y \cup Z$ where $Y$ and $Z$ are closed subsets of $\mathbb{A}^n$. Then Hilbert's Nullstellensatz gives us non-zero radical ideals $I, J$, such that $Y = \mathscr{I}(I)$ and $Z = \mathscr{I}(J)$. Then we have $\mathscr{I}(IJ) = \mathbb{A}^n$ and in particular the product of $I$ and $J$ is a subset of $(0)$. This cannot be the case, because the product of two non-zero ideals cannot be $(0)$ in $\mathbb{C}[x_1, \dots, x_n]$.

A topological space is called *connected*, if it cannot be written as the union of two disjoint non-empty closed subsets. A subset $C$ of a topological space $X$ is called *connected* if $C$ is connected as the topological spaced endowed with the induced topology from $X$. A maximal connected subset of $X$ is called a *connected component* of $X$.

**1.3 Lemma.** Connected components of a topological space are closed.

*Proof.* Let $C$ be a connected component of a topological space $X$. If $\overline{C}$ is connected we are done because $C$ cannot be a proper subset of a connected subset of $X$. Assume towards a contradiction that $\overline{C}$ is not connected. Then there exist closed subsets $U, V \subseteq X$ such that $\overline{C} = U \cup V$, $\overline{C} \cap U \cap V = \emptyset$ and $\overline{C}$ is not a subset of $U$ nor of $V$. Since $C$ is connected it has an empty intersection with either $U$ or $V$. Thus, $C$ is a subset of $U$ or $V$ which are proper closed subsets of $\overline{C}$. This is a contradiction to the definition of $\overline{C}$ because $\overline{C}$ is the smallest closed subset of $X$ containing $C$. □

Let $X$ and $Y$ be two topological spaces. A map $f : X \to Y$ is called *continuous*, if the preimage of any open subset in $Y$ is an open subset in $X$.

**1.4 Lemma.** Let $f : X \to Y$ be a continuous map between two topological spaces $X$ and $Y$. If $C \subseteq X$ is connected, then the image $f(C) \subseteq Y$ is connected as well.

*Proof.* Assume towards a contradiction that $f(C)$ is not connected. Thus, there are two non-empty closed subsets $U, V \subseteq Y$, with $f(C) \subseteq U \cup V$ and $U \cap V \cap f(C) = \emptyset$. Consequently, $f^{-1}(U)$ and $f^{-1}(V)$ are non-empty closed subsets of $X$ such that $C \cap f^{-1}(U) \cap f^{-1}(V) = \emptyset$ and $C \subseteq f^{-1}(U) \cup f^{-1}(V)$. This contradicts the assumption that $C$ is connected. □

A topological space is called *noetherian* if every descending chain of closed subsets $Z_1 \supseteq Z_2 \supseteq \dots$ becomes stationary. This means, that there is an index $s$ such that $Z_s = Z_{s+i}$ for all $i > 0$.

**1.5 Lemma.** Let $Y$ be a non-empty closed subset of a noetherian topological space $X$. Then it is possible to write $Y$ as a finite union of closed irreducible subsets, say $Y = Z_1 \cup \dots \cup Z_m$, such that $Z_i \nsubseteq Z_j$ whenever $i \neq j$. The collection of closed irreducible sets $\{Z_1, \dots, Z_m\}$ is uniquely determined by $Y$.

*Proof.* [15, Proposition 1.19] Our first goal is to show that every non-empty $Y$ can be written as a finite union of closed irreducible subsets. We assume towards a contradiction that this is not the case for some non-empty subsets. Let $\mathscr{K}$ be the set of all non-empty closed subsets of $X$ that cannot be written as a finite union of closed irreducible subsets. Since $X$ is noetherian, $\mathscr{K}$ has a minimal element since otherwise we would obtain a non-stationary descending chain. Let $Y' \in \mathscr{K}$ be one of those minimal elements of $\mathscr{K}$.

Because $Y'$ is reducible, we can choose two proper closed subsets $Y_1$ and $Y_2$ that satisfy $Y' = Y_1 \cup Y_2$. These $Y_1$ and $Y_2$ cannot be in $\mathscr{K}$ and thus can be written as a finite union of closed irreducible subsets. This implies, that the same holds for $Y'$ and hence, $\mathscr{K}$ has to be the empty set.

Let $Z_1 \cup \cdots \cup Z_m$ be a decomposition of $Y$ into closed irreducible subsets. Without loss of generality, we can assume $Z_i \nsubseteq Z_j$ for $i \neq j$, because we can omit those $Z_i$ that are contained in some $Z_j$ with $i \neq j$.

Now suppose that we have $Y = Z_1 \cup \cdots \cup Z_m = Z'_1 \cup \cdots \cup Z'_n$, where all $Z'_i \subseteq Y$ are closed and irreducible. For $i \in \{1, \ldots, m\}$ we have $Z_i = \bigcup_{j=1}^{n} (Z_i \cap Z'_j)$. Since $Z_i$ is irreducible, it cannot be the finite union of more than one closed subsets. Thus, there exists $\nu(i) \in \{1, \ldots, n\}$ with $Z_i \subseteq Z'_{\nu(i)}$. Similarly, we can do the same construction for $j \in \{1, \ldots, n\}$ and obtain $\mu(j) \in \{1, \ldots, m\}$, such that $Z'_j \subseteq Z_{\mu(j)}$. Thus, $Z_i \subseteq Z_{\mu(\nu(i))}$ which implies $i = \mu(\nu(i))$. Similarly, we have $j = \nu(\mu(j))$ for all $j \in \{1, \ldots n\}$. This implies, that $m = n$ that $\mu$ and $\nu$ are mutually inverse permutations of $\{1, \ldots, n\}$, and that $Z_i = Z_{\nu(i)}$ for all $i$. So up to permutation of the indices, the two decompositions are the same. $\square$

A ring $R$ is called *noetherian*, if every ascending chain of ideals $I_1 \subseteq I_2 \subseteq \ldots$ becomes stationary, i.e. there exists $s \in \mathbb{N}$ such that $I_s = I_{s+1} = \ldots$.

**1.6 Theorem** (Hilbert's basis theorem)**.** Let $R$ be a noetherian ring. Then $R[x]$ is noetherian.

**1.7 Remark.** Hilbert's basis theorem shows that $\mathbb{C}[x_1, \ldots, x_n]$ is noetherian. By Hilbert's Nullstellensatz radical ideals of $\mathbb{C}[x_1, \ldots, x_n]$ correspond to closed subsets in $\mathbb{A}^n$. Hence, the Zariski topology on $\mathbb{A}^n$ is notherian because we can translate an increasing chain of radical ideas into a decreasing chain of closed subsets. Thus, every closed subset of $\mathbb{A}^n$ can be written as a union of finitely many closed irreducible subsets by Lemma 1.5. It is further possible using Theorem 1.6 to show that every closed subset of $\mathbb{A}^n$ can be written as $\mathscr{Z}(S)$, where $S$ is a finite subset of $\mathbb{C}[x_1, \ldots, x_n]$.

## 1.2 Affine varieties

**1.8 Definition.** An *affine variety* is a closed subset of $\mathbb{A}^n$ for some $n \in \mathbb{N}_1$. A *quasi-affine variety* is a non-empty open subset of an affine variety. A closed subset of a quasi-affine $X$ variety is called a *closed subvariety of* $X$ and has the induced topology of the quasi-affine variety $X$.

**1.9 Example.** Let $p \in \mathbb{C}[z]$ be a polynomial. The *Danielewski surface* $D_p$ is defined as $\{(x, y, z) \in \mathbb{A}^3 \mid xy = p(z)\}$. It is obvious, that $D_p$ is an affine variety, because it is the set $\mathscr{I}(xy - p(z)) \subseteq \mathbb{A}^3$.

A subset of an affine variety is called *locally closed*, if it is the intersection of a closed and an open subset. If a subset of an affine variety is the finite union of locally closed subsets, it is called *constructible*.

**1.10 Remark.** (1) Finite union, finite intersection and complement of constructible sets are again constructible.
(2) At the beginning of [5, Section 4.4] it is noticed, that if a subset $S$ of an affine variety $X$ is constructible, then $S$ contains a set $U$ which is open and dense in $\overline{S}$.

**1.11 Definition.** Let $R$ be a ring and $R'$ an $R$-module. An element $a \in R'$ is said to be *integral* over $R$, if it is the root of a monic polynomial in $R[x]$, i.e.

$$a^n + r_1 a^{n-1} + \cdots + r_n = 0, \text{ where } r_1, \ldots, r_n \in R.$$

We say that $R'$ is *integral* over $R$, if every element $a \in R'$ is integral over $R$. If no element $x \in R' \setminus R$ is integral over $R$, $R$ is called *integrally closed* in $R'$.

**1.12 Remark.** Let $R$ be a ring and $R'$ an $R$-module. It can be shown that $a \in R'$ is integral over $R$, if and only if $R[a]$ is finitely generated $R$-module.

An affine variety $X$ is *normal* if the coordinate ring $\mathbb{C}[x_1, \ldots, x_n]/\mathscr{I}(X)$ is integrally closed in its field of fractions. Here the field of fractions of a ring means the smallest field, that contains the ring.

**1.13 Example.** (1) The affine space $\mathbb{A}^n$ is normal. This follows, because its coordinate ring is isomorphic to $\mathbb{C}[x_1, \ldots, x_n]$ which is integrally closed in $\mathbb{C}(x_1, \ldots, x_n)$.
(2) The curve defined by $\{(x, y) \in \mathbb{A}^2 \mid y^2 = x^3\}$ is not normal. This follows since $\mathbb{C}[x, y]/(y^2 - x^3) \cong \mathbb{C}[t^2, t^3]$ and the field of fractions of $\mathbb{C}[t^2, t^3]$ is $\mathbb{C}(t)$. Thus, there is $t \in \mathbb{C}(t) \setminus \mathbb{C}[t^2, t^3]$ which is integral over $\mathbb{C}[t^2, t^3]$.

Let $Y \subseteq \mathbb{A}^n$ be a quasi-affine variety. A function $f : Y \to \mathbb{C}$ is called *regular at point* $P \in Y$, if there is an open subset $U \subseteq Y$ that contains $P$ and polynomials $g, h \in \mathbb{C}[x_1, \ldots, x_n]$ with $h \neq 0$ on $U$, such that $f_{|U} = g/h$ as functions on $U$. A function $f : Y \to \mathbb{C}$ is called *regular*, if $f$ is regular at all points of $Y$. We denote the $\mathbb{C}$-algebra of regular functions on $Y$ by $\mathcal{O}(Y)$.

**1.14 Lemma.** Let $X$ be a quasi-affine variety. If $f$ and $g$ are regular functions on $X$ that restrict to the same function on a dense subset $D \subseteq X$ then $f = g$.

*Proof.* [15, Lemma 2.2 ii)] The set $Z = \{P \in X \mid f(P) = g(P)\}$ is the preimage of $\{0\} \subseteq \mathbb{C}$ under the regular function $g - f : X \to \mathbb{C}$. Because $\{0\} \subseteq \mathbb{C}$ is a closed subset, $Z$ is closed as well. Since $D$ is a subset of $Z$ which is dense in $X$, we have $X = \overline{D} \subseteq Z$. Thus, $g - f$ has to be the constant zero function and in particular $g = f$. □

In [5, Section 1.5] it is noted that the $\mathbb{C}$-algebra of regular functions $\mathcal{O}(X)$ of an affine variety $X$ is isomorphic to $\mathbb{C}[x_1, \ldots, x_n]/\mathscr{I}(X)$. Note that by Remark 1.7 $\mathscr{I}(X)$ can always be written as an ideal generated by finitely many functions from $\mathbb{C}[x_1, \ldots, x_n]$.

Let $X$ and $Y$ be two quasi-affine varieties. Then a *morphism* from $X$ *to* $Y$ is a continuous map $\varphi : X \to Y$ such that for every open subset $U \subseteq Y$ and every regular function $f \in \mathcal{O}(U)$, the function $f \circ \varphi : \varphi^{-1}(U) \to \mathbb{C}$ is regular on $\varphi^{-1}(U) \subseteq X$.

A morphism is called an *isomorphism*, if the inverse map is still a morphism. If there exists an isomorphism between two varieties, they are called *isomorphic*. An isomorphism from a variety to itself is called an *automorphism*. The set of all automorphisms of a variety $X$ is denoted by $\mathrm{Aut}(X)$ and equipped with the composition, it has the structure of a group.

**1.15 Remark.** (1) Let $\varphi : X \to Y$ be a morphism. Then we can get a homomorphism of $\mathbb{C}$-algebras by

$$\varphi^* : \mathcal{O}(X) \to \mathcal{O}(Y), f \mapsto f \circ \varphi.$$

(2) Let $C \subseteq Y$ be a locally closed subset. Since the preimages of open and closed subsets under a morphism are open and closed respectfully, $f^{-1}(C)$ is locally closed.
(3) Define $\varphi : \mathbb{A}^2 \to \mathbb{A}^2, (x, y) \mapsto (x, xy)$. Note that this is a morphism. The image $\mathrm{Im}(\varphi) = \mathbb{A}^2 \setminus \{(0, c) \mid c \neq 0\}$ is not locally closed in $\mathbb{A}^2$. This follows, because $\mathbb{A}^2 \setminus \{(0, c) \mid c \neq 0\}$ is not open and has the closure $\mathbb{A}^2$.

**1.16 Theorem** (Chevalley's Theorem)**.** If $\varphi : X \to Y$ is a morphism of affine varieties, then the image of a constructible subset is constructible.

We use the following lemma in the proof of Lemma 3.11.

**1.17 Lemma.** The $\mathbb{C}$-algebra $\mathcal{O}(\mathbb{C}^*)$ is isomorphic to $\mathbb{C}[t, t^{-1}]$ in a variable $t$.

*Proof.* The variety $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ is isomorphic to $Y := \mathscr{Z}(xy - 1) \subseteq \mathbb{C}^2$. Indeed, $Y = \{(x, x^{-1} \mid x \in \mathbb{C}^*\}$ and the map $f : \mathbb{C}^* \to Y, x \mapsto (x, x^{-1})$ is a morphism. Moreover, the inverse map $g : Y \to \mathbb{C}^*, (x, x^{-1}) \mapsto x$ is a morphism. Therefore, $Y$ is isomorphic to $\mathbb{C}^*$ and thus $\mathcal{O}(\mathbb{C}^*) = \mathcal{O}(\mathscr{Z}(xy-1)) = \mathbb{C}[x, y]/(xy-1) \cong \mathbb{C}[t, t^{-1}]$. □

## 1.3 Dimension of affine varieties

**1.18 Definition.** The *dimension* of a non-empty topological space $X$ is the supremum of the integers $r$ for which there exists a chain $Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_r$ of closed irreducible subsets. If the chain can be arbitrarily long, we say that $\dim(X) = \infty$. Let $n \in \mathbb{N}$. Then the $n$-dimensional *torus* is defined as $(\mathbb{C}^*)^n$.

The *height* $ht(p)$ of a prime ideal $p$ is the supremum of integers, such that there exists a chain of prime ideals

$$p_0 \subsetneq p_1 \subsetneq \cdots \subsetneq p_r = p$$

ending in $p$. The *Krull dimension* of a commutative ring $R$ is the supremum of the heights of prime ideals in $R$. If there exists arbitrary long chains of prime ideals, we say that the height is infinite. We denote the Krull dimension of a ring $R$ by $\mathrm{Kdim}(R)$.

**1.19 Remark.** (1) If a ring is noetherian, every prime ideal of this ring has a finite height.
(2) There is only one prime ideal in a field. Thus, each field has Krull dimension 0.
(3) There is an example of a noetherian ring which has infinite Krull dimension.

**1.20 Example.** There exists a noetherian topological space of infinite dimension. Indeed, let $X = \mathbb{N}$ and define the closed subsets of $X$ by $Y_n := \{0, \ldots, n\}$ for $n \in \mathbb{N}$. This topological space is obviously noetherian, because every chain can at most have length $n$. But the chains can become arbitrary long, thus the dimension of $X$ is infinite.

Before we state further lemmas about the dimension of a variety, we recall properties of the Krull dimension.

**1.21 Theorem.** If $R$ is a noetherian ring, then $\mathrm{Kdim}(R[x]) = \mathrm{Kdim}(R) + 1$.

*Proof.* This is proven in [1, solution of Exercise 21.40]. □

Since $\mathbb{C}$ is a field, the Krull dimension of $\mathbb{C}$ is 0. Thus $\mathrm{Kdim}(\mathbb{C}[x_1, \ldots, x_n]) = n$. Recall that prime ideals in $\mathbb{C}[x_1, \ldots, x_n]$ correspond to closed irreducible subsets of $\mathbb{A}^n$. Thus, the Krull dimension of $\mathbb{C}[x_1, \ldots, x_n]$ coincides with the dimension of the affine space $\mathbb{A}^n$ endowed with the Zariski topology.

**1.22 Lemma.** Let $R$ be an finitely generated $\mathbb{C}$-algebra without zero divisors.

(1) The Krull dimension of $R$ is finite.

(2) Let $I = (a_1, \ldots, a_r)$ be an ideal generated by $r$ elements. If $p$ is a minimal prime ideal that contains $I$, we have $ht(p) \leq r$.

(3) If $q \subseteq p$ are prime ideals of $R$ and

$$q = p_0 \subsetneq p_1 \subsetneq \cdots \subsetneq p_r = p$$

is any maximal chain of prime ideals from $q$ to $p$, we have $r = ht(p) - ht(q)$.

*Proof.* (1) is proven in [1, Theorem 21.4], (2) is proven in [1, Corollary 21.7] and (3) is proven in [1, solution of Exercise 21.27]. □

The results from the lemma above can be geometrically translated into the following statement.

**1.23 Lemma.** Let $Z, Y$ be non-empty irreducible affine varieties.

(1) The dimension of $Z$ is finite.

(2) If the ideal $\mathscr{I}(Y) \subseteq \mathbb{C}[x_1, \ldots, x_n]$ can be generated by $r$ elements, we have $\dim(Y) \geq n - r$.

(3) If $Z \subseteq Y$ and $Z = Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_r = Y$ is any maximal chain of closed irreducible subvarieties, we have $r = \dim(Y) - \dim(Z)$.

The number $r$ in Lemma 1.23 (3) is the *codimension* of $Z$ in $Y$ and denoted by $\mathrm{codim}_Y(Z)$.

**1.24 Remark.** For an affine variety $X$ and a point $a \in X$ we obtain the following:

$$\operatorname{codim}_X\{a\} = \max\{\dim X_i : a \in X_i\},$$

where the $X_i$ are the irreducible components of $X$.

**1.25 Lemma.** Let $X$ be an irreducible affine variety of dimension 1. Then a proper closed subset is finite.

*Proof.* Let $C$ be a proper closed subset of $X$. By Lemma 1.5 we know that $C$ is the union of finitely many closed irreducible subsets of $\mathbb{C}$. Since $X$ has dimension one, all proper closed irreducible subsets must be of dimension 0, i.e. are single points. Thus, $C$ is the union of finitely many points. $\square$

Let $f \in \mathbb{C}[x_1, \dots, x_n]$. We denote by $\frac{\partial f}{\partial x_i}$ the formal partial derivative of $f$ by $x_i$.

**1.26 Definition.** Let $X \subseteq \mathbb{A}^n$ be an affine variety with $\mathscr{I}(X) = (f_1, \dots, f_r)$ and let $a \in X$ be a point. We call $a \in X$ *smooth*, if the rank of the matrix

$$\left(\frac{\partial f_i}{\partial x_j}(a)\right)_{i,j}$$

is at least $n - \operatorname{codim}_X\{a\}$. We call $X$ *smooth*, if all points $x \in X$ are smooth.

**1.27 Lemma.** Let $p$ be a polynomial in one variable. Then $D_p$ is smooth if and only if $p$ has simple roots, i.e. all its roots are different from each other.

*Proof.* By definition we have $\mathscr{I}(D_p) = (xy - p(z))$ and $D_p \subseteq \mathbb{A}^3$. Thus, we have the following matrix

$$\left(\left(\frac{\partial(xy - p(z))}{\partial x}\right), \left(\frac{\partial(xy - p(z))}{\partial y}\right), \left(\frac{\partial(xy - p(z))}{\partial z}\right)\right) = (y, x, -p'(z)).$$

This implies that only the points of $D_p$ that fulfill $xy - p(z) = y = x = p'(z) = 0$ are not smooth. If $p$ has simple roots, the system $p(z) = p'(z) = 0$ has no solution and $D_p$ is smooth. If $p$ has not only simple roots, there is $z_0 \in D_p$ with $p(z_0) = p'(z_0) = 0$ and $(0, 0, z_0) \in D_p$ is not smooth. $\square$

**1.28 Remark.** It can be shown [15, Theorem 2.8] that every smooth affine variety is a normal affine variety.

## 1.4 Affine algebraic groups

**1.29 Definition.** An *affine algebraic group* is an affine variety $G$ that is equipped with the structure of a group such that the inverse map $i : G \to G, g \mapsto g^{-1}$ and the group multiplication $m : G \times G \to G, (g, h) \mapsto g \cdot h$ are morphisms. A closed subgroup $H$ of $G$ is an *algebraic subgroup* with the induced topology of $G$.

**1.30 Example.** (1) Two important algebraic groups are $\mathbb{C}^+$ which is the *additive group* of the field $\mathbb{C}$, and $\mathbb{C}^*$ which is the *multiplicative group* of the field $\mathbb{C}$.
(2) An *algebraic torus* is an affine algebraic group that is isomorphic to $(\mathbb{C}^*)^n$, where $n \in \mathbb{N}$.

**1.31 Lemma.** The automorphism group of $\mathbb{C}^+$ is isomorphic to $\mathbb{C}^*$.

*Proof.* Let $\varphi$ be an automorphism of $\mathbb{C}^+$. First, we consider the image of 1 which we denote by $c$. This $c$ has to be from $\mathbb{C}^*$, because $\varphi$ would not be injective otherwise.

Every $k \in \mathbb{N}$ can be written as $1 + \cdots + 1$ and thus $\varphi(k) = \varphi(1 + \cdots + 1) = ck$. By Lemma 1.25, $\mathbb{N}$ is a dense subset of $\mathbb{C}^+$, because it is an infinite subset of the one-dimensional variety $\mathbb{C}^+$. By Lemma 1.14 we obtain our claim, because $\varphi$ and $f : \mathbb{C}^+ \to \mathbb{C}^+, x \mapsto cx$ coincide on the dense subset $\mathbb{N} \subseteq \mathbb{C}^+$. □

**1.32 Theorem.** Let $G$ be an irreducible affine algebraic group of dimension 1. Then $G$ is isomorphic to $\mathbb{C}^+$ or to $\mathbb{C}^*$.

*Proof.* This is proven in [5, Theorem in Chapter 20.5]. □

Further examples of linear algebraic groups include the group of all invertible $n \times n$-matrices $\mathrm{GL}_n$ and the group of all $n \times n$-matrices of $\mathrm{GL}_n$ with determinant 1 $\mathrm{SL}_n$.

**1.33 Lemma.** The general linear group $\mathrm{GL}_n$ and the special linear group $\mathrm{SL}_n$ are affine algebraic groups.

*Proof.* The first step is to show that both $\mathrm{GL}_n$ and $\mathrm{SL}_n$ are affine varieties. Consider first the closed subset $S = \{(A, t) \in \mathrm{Mat}_{n\times n} \times \mathbb{A}^1 \mid \det(A) \cdot t = 1\} \subseteq \mathrm{Mat}_{n\times n} \times \mathbb{A}^1$ where $\det(A)$ is a regular function in $\mathcal{O}(\mathrm{Mat}_{n\times n})$. Now the projection onto the first coordinate $S \to \mathrm{GL}_n, (A, t) \mapsto A$ gives an isomorphism of $S$ and $\mathrm{GL}_n$ since the inverse map is a morphism. Thus, $\mathrm{GL}_n \cong \mathscr{Z}(\det(A)t - 1) \subseteq \mathrm{Mat}_{n\times n} \times \mathbb{A}^1 \cong \mathbb{A}^{2^n+1}$.

Similarly $\mathrm{SL}_n = \{A \in \mathrm{Mat}_{n\times n} \cong \mathbb{A}^{n^2} \mid \det(A) = 1\} \subseteq \mathrm{Mat}_{n\times n}$ is a closed subset.

The multiplication of two matrices in $\mathrm{Mat}_{n\times n}$ is given by regular functions on $\mathrm{Mat}_{n\times n}$. The inverse for a matrix $A \in \mathrm{GL}_n$ is equal to $\det(A)^{-1}$ times the adjoint matrix. Indeed $\det(A)^{-1} \in \mathcal{O}(\mathrm{Mat}_{n\times n})$ and the map $\mathrm{Mat}_{n\times n} \to \mathrm{Mat}_{n\times n}$ which sends a matrix to its adjoint matrix is a morphism. Hence, the map $\mathrm{GL}_n \to \mathrm{GL}_n, A \mapsto A^{-1}$ is also a morphism. The proof follows. □

**1.34 Definition.** Let $G$ be a group. An element $g \in G$ is called *divisible* if for each $n \in \mathbb{N}_1$ there exists $h \in G$ such that $g = h^n$.

**1.35 Example.** (1) Every element $g \in \mathbb{C}^*$ is divisible, because $g = h^n$ has a solution in $\mathbb{C} \setminus \{0\}$ for each $n \in \mathbb{N}_1$.
(2) No non-trivial element in a finite group is divisible. This follows, because each element to the power of the group order is 1.

Consider the quotient $\mathrm{SL}_2/T$ of the group $\mathrm{SL}_2$ by the torus

$$T := \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mid t \in \mathbb{C}^* \right\}.$$

For the equivalence classes in $\mathrm{SL}_2/T$ we use the notation

$$\left[ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right] = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mid t \in \mathbb{C}^* \right\}.$$

We now consider the structure of $\mathrm{SL}_2/T$. Recall that two elements $A, B \in \mathrm{SL}_2$ are in the same equivalence class if there exists $t \in T$ such that $A = Bt$. Let $\mathcal{A} := \{(ab, cd, ad) \in \mathbb{A}^3 \mid a, b, c, d \in \mathbb{C} \text{ and } ad - cb = 1\}$ and

$$\pi : \mathrm{SL}_2/T \to \mathcal{A}, \left[ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right] \mapsto (ab, cd, ad).$$

To check if this map is well defined, we show $\pi(A) = \pi(At)$ for all $A \in \mathrm{SL}_2$ and $t \in T$. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2$ and $t = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \in T$, then we have

$$\pi(At) = \pi \begin{pmatrix} ta & t^{-1}b \\ tc & t^{-1}d \end{pmatrix} = (tat^{-1}b, tct^{-1}d, tat^{-1}d) = (ab, cd, ad) = \pi(A).$$

Thus, $\pi$ is well defined. Since $(ab, cd, ad) \in \mathcal{A}$ has the preimage $\left[ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right]$ the map $\pi$ is surjective. Now we check if $\pi$ is injective. Let $A = \left[ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right]$ and $A' = \left[ \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \right]$ such that $(ab, cd, ad) = \pi(A) = \pi(A') = (a'b', c'd', a'd')$.
Assume $a \neq 0$ and $a' \neq 0$ first. Then there is $z \in \mathbb{C}^*$ with $a = za'$. Thus, $b = z^{-1}b'$ and $d = z^{-1}d'$ because $ab = a'b'$ and $ad = a'd'$. Moreover,

$$ad - cb = 1 = a'd' - c'b' \Leftrightarrow za'z^{-1}d' - cz^{-1}b' = a'd' - c'b' \Leftrightarrow c = z^{-1}c'.$$

Thus, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \cdot \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}$.
Now assume $a = a' = 0$. Then $c, c' \in \mathbb{C}^*$ since $ad - cb = 1 = a'd' - c'b'$. Thus, there exists $z \in \mathbb{C}^*$ with $c = zc'$. Moreover, $d = z^{-1}d'$ because $cd = c'd'$. This means that

$$ad - cb = 1 = a'd' - c'b' \overset{a=a'=0}{\Leftrightarrow} cb = c'b' \Leftrightarrow zc'b = c'b' \Leftrightarrow b = z^{-1}b'.$$

Moreover, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \cdot \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}$.
Finally, we assume $a \in \mathbb{C}^*$ and $a' = 0$ which is analogous to $a' \in \mathbb{C}^*$ and $a = 0$. Then we have $b = 0$ because $ab = a'b'$. Thus, in particular $ad = 1$. This gives us a contradiction because $ad = 1 \neq 0 = a'd'$.
This shows that if $(ab, cd, ad) = (a'b', c'd', a'd')$ we have $A = A'$ and $\pi$ is injective. Thus, we have the following description for the elements in $\mathrm{SL}_2/T$:

$$\{(ab, cd, ad) \in \mathbb{A}^3 \mid a, b, c, d \in \mathbb{C} \text{ and } ad - cb = 1\}. \tag{1}$$

**1.36 Lemma.** Let $p = z^2 - z$. Then the two varieties $D_p$ and $\mathrm{SL}_2/T$ are isomorphic.

*Proof.* By (1), $\mathrm{SL}_2/T$ is isomorphic to $X = \{(ab, cd, ad) \in \mathbb{A}^3 \mid a, b, c, d \in \mathbb{C}, ad - cb = 1\}$. Thus, we show that $X$ is isomorphic to $\{(x, y, z) \in \mathbb{A}^3 \mid xy = z^2 - z\}$. Let $(ab, cd, ad) \in X$, then we have

$$xy = (ab)(cd) = (ad)(cb) = (ad)(ad - 1) = (ad)^2 - (ad) = z^2 - z,$$

because $ad - cb = 1 \Leftrightarrow cb = ad - 1$. Thus, the map $X \to D_p, (ab, cd, ad) \mapsto (x, y, z)$ is an isomorphism of varieties. $\square$

A *regular group action* of an affine algebraic group $G$ on an affine variety $S$ is a morphism of affine varieties $G \times S \to S$ which satisfies $1 \cdot s = s$ and $(gh) \cdot s = g \cdot (h \cdot s)$ for all $g, h \in G$ and $s \in S$. The *orbit of* $s \in S$ is the set $G \cdot s = \{g \cdot s \mid g \in G\} \subseteq S$. The *stabilizer* of an element $s \in S$ is defined as $\mathrm{Stab}_G(s) := \{g \in G \mid g \cdot s = s\}$. The action is called *trivial*, if the stabilizer of every element is the entire group. An action is *faithful* if for each two elements $h, g \in G$ there exists $x \in S$, such that $h \cdot x \neq g \cdot x$. We call an action *free* if $h \cdot x = g \cdot x$ for $x \in X$ and $g, h \in G$ always implies $h = g$.

**1.37 Example.** Let $H, G$ be two algebraic subgroups of an algebraic group $K$. Then we define the *conjugation of* $h \in H$ *with* $g \in G$ as $ghg^{-1}$. This is a morphism and if $gHg^{-1} := \{ghg^{-1} \mid h \in H\} \subseteq H$, it is a group action. In this case we say that $H$ *is normalized by* $G$.

**1.38 Definition.** Let $X$ be an affine variety of dimension $n$. We call $X$ *toric*, if $T \cong (\mathbb{C}^*)^n$ acts non-trivially, faithfully and regularly on $X$.

**1.39 Lemma.** Let $G$ be an affine algebraic group that acts regularly on an affine variety $S$. For $s \in S$ it follows that

$$G \cdot s \cong G/\mathrm{Stab}_G(s)$$

as quasi-affine varieties.

*Proof.* This is proven in [13, Corollary 7.13]. $\square$

In the last part of this section we closely follow [14].

**1.40 Lemma.** Let $U$ and $V$ be open dense subsets of an algebraic group $G$. Then $G = U \cdot V$.

*Proof.* Let $x \in G$. Then $x \cdot V^{-1} := \{xv^{-1} \mid v \in V\}$ and $U$ are dense and open subsets of $G$. If the intersection of $x \cdot V^{-1}$ and $U$ is empty, the subset $U \subseteq G$ would be a subset of the complement of $x \cdot V^{-1}$. This is a contradiction because the dense subset $U$ cannot be a subset of a proper closed subset $G \setminus X \cdot V^{-1}$. Thus, $X \cdot V^{-1}$ and $U$ have a non-empty intersection, i.e. $xv^{-1} = u$ for some $v \in V, u \in U$. So $x$ can be written as the product of an element of $V$ and $U$, and in particular $x \in U \cdot V$. $\square$

**1.41 Lemma.** Let $H \leq G$ be a subgroup of an algebraic group $G$. Then

(1) if $H$ is constructible, then $H$ coincides with its closure $\overline{H}$,

(2) if $H$ contains a dense open subset of its closure $\overline{H}$, then $H = \overline{H}$,

(3) if $H$ is locally closed, then it is closed.

*Proof.* First we consider (1) and (2). By Remark 1.10 a constructible subset $H \leq G$ contains a subset $U$ which is dense and open in $\overline{H}$. Then $H$ is open in $\overline{H}$, because it is the union of $\bigcup_{h \in H} h \cdot U$, where $h \cdot U \subseteq \overline{H}$ are open. Hence, by Lemma 1.40 $\overline{H} = H \cdot H = H$. (3) directly follows from (2). □

**1.42 Lemma.** Let $G$ be an algebraic group which acts regularly on an affine variety $X$. Then, the orbit $G \cdot x$ is a locally closed subvariety of $X$. Moreover, the closure $\overline{G \cdot x}$ is the union of $G \cdot x$ and of orbits of strictly smaller dimension. Any orbit of minimal dimension in $\overline{G \cdot x}$ is closed, in particular, the closure $\overline{G \cdot x}$ of $G \cdot x$ contains a closed orbit.

*Proof.* Consider the map

$$\rho_x : G \to X, g \mapsto g \cdot x.$$

This map is a morphism and by Theorem 1.16 the image $\mathrm{Im}(\rho_x) = G \cdot x$ is a constructible subset of $X$. By Remark 1.10 there exists a subset $U \subseteq G \cdot x$ which is open and dense in $\overline{G \cdot x}$. Since $G$ acts transitively on $G \cdot x$, the set $G \cdot x = \bigcup_{g \in G} g \cdot U$ is open in $\overline{G \cdot x}$. Thus, $G \cdot x \subseteq X$ is locally closed.

Since $G \cdot x$ is dense open subset of $\overline{G \cdot x}$ we have that $\overline{G \cdot x} \setminus G \cdot x$ has dimension strictly smaller than $\overline{G \cdot x}$. Furthermore, $\overline{G \cdot x}$ is a $G$-subvariety of $X$, thus $\overline{G \cdot x} \setminus G \cdot x$ is a union of orbits of $G$.

Finally, we have to prove that any $G$-orbit $O$ of minimal dimension in $\overline{G \cdot x}$ is closed. Assume that there is one that is not closed. Then the closure $\overline{O} \subseteq \overline{G \cdot x}$ contains an orbit of smaller dimension which is a contradiction. □

**1.43 Definition.** An element $g$ of an algebraic group $G$ is called *unipotent*, if the closure of the group generated by $g$ is isomorphic to the additive group $\mathbb{C}^+$. An algebraic group $G$ is called *unipotent*, if each element of $G$ is unipotent.

If a group $G$ is itself isomorphic to $\mathbb{C}^+$, it is obviously unipotent.

**1.44 Theorem.** A unipotent group $G$ acts on an affine variety $X$ with closed orbits, i.e. a $G$-orbit of an element $x \in X$ is closed in $X$.

*Proof.* This is proven in [19, Theorem 2]. □

# 2 Ind-varieties and ind-groups

The notion of an ind-group goes back to Shafarevich [22], who calls these objects infinite dimensional algebraic groups. An automorphism group $\mathrm{Aut}(X)$ of an affine variety $X$ has a natural structure of an ind-group (see Theorem 2.6). Our first step is to consider ind-varieties.

**2.1 Definition.** An *ind-variety* is a set V together with an ascending filtration $V_0 \subseteq V_1 \subseteq \ldots \subseteq V$ for which the following must be satisfied:

(1) $V = \bigcup_{k \in \mathbb{N}} V_k$;

(2) each $V_k$ has the structure of an affine variety;

(3) for all $k \in \mathbb{N}$ the subset $V_k \subseteq V_{k+1}$ is closed in the Zariski topology.

A *morphism* between ind-varieties $V = \bigcup_k V_k$ and $W = \bigcup_k W_k$ is a map $\phi : V \to W$ such that for each $k$ there is $l \in \mathbb{N}$, such that $\phi(V_k) \subseteq W_l$ and such that the induced map $V_k \to W_l$ is a morphism of algebraic varieties. A morphism is called an *isomorphism* of ind-varieties if it has an inverse map that is an morphism as well.

An ind-variety $V = \bigcup_k V_k$ has a natural topology: $S \subseteq V$ is closed, respectively open, if $S_k := S \cap V_k \subseteq V_k$ is closed respectively open for all $k$. Obviously, a closed subset $S \subseteq V$ has the natural structure of an ind-variety which is called an *ind-subvariety.*

Let $G$ be an ind-variety with the filtration $\bigcup_i G_i$. Then the direct product $G \times G$ has the filtration $\bigcup_i (G_i \times G_i)$ because $G_i \times G_i \subseteq G_{i+1} \times G_{i+1}$ is closed.

**2.2 Definition.** An ind-variety $G$ is called an *ind-group* if the underlying set $G$ is a group and the map $G \times G \to G, (g,h) \mapsto gh^{-1}$ is a morphism of ind-varieties.

For an ind-group $G$, we can also consider subgroups $H$ of $G$. If $H$ is closed, $H$ is an ind-group under the ind-subvariety structure on $G$ as well. Similar to varieties, a closed subgroup $H$ of an ind-group $G$ is called an *algebraic subgroup*, if $H$ is an algebraic subset of $G$. That is, $H$ is a closed subset of some $G_i$, where $G_1 \subseteq G_2 \subseteq \ldots$ is a filtration of $G$. In this thesis, we consider an endomorphism $f \in \mathrm{End}(\mathbb{A}^n)$ as

$$f : \mathbb{A}^n \to \mathbb{A}^n, (a_1, \ldots, a_n) \mapsto (f_1(a_1, \ldots, a_n), \ldots, f_n(a_1, \ldots, a_n)),$$

where $f_1, \ldots, f_n \in \mathbb{C}[x_1, \ldots, x_n]$. Hence, we use the notation $f = (f_1, \ldots, f_n)$.

**2.3 Example.** (1) The set $\mathrm{End}(\mathbb{A}^n)$ of the endomorphisms of $\mathbb{A}^n$ has the structure of an ind-variety. The filtration can be given by vector spaces

$$\mathrm{End}(\mathbb{A}^n)_d = \{f = (f_1, \ldots, f_n) \in \mathrm{End}(\mathbb{A}^n) \mid \deg f := \max_i f_i \leq d\}.$$

(2) One can prove that $\mathrm{Aut}(\mathbb{A}^n)$ with the following filtration $\bigcup_d (\mathrm{Aut}(\mathbb{A}^n) \cap \mathrm{End}(\mathbb{A}^n)_d)$ is an ind-group (see [4]).
(3) Let $V$ be a $\mathbb{C}$-vector space of countable dimension. Then we can choose a filtration $V_1 \subseteq V_2 \subseteq \ldots$ of finite dimensional subspaces. This gives us the structure of

an ind-variety on $V$. Note that two filtrations by finite dimensional subspaces of $V$ are always equivalent.

With those examples, we consider more general constructions.

**2.4 Lemma** (Stampfli)**.** Let $X$ and $Y$ be affine varieties. Then the set of morphisms $\mathrm{Mor}(X, Y)$ has a canonical structure of an ind-variety.

*Proof.* [23, Lemma 3.8] Let $n \in \mathbb{N}$ such that $Y \subseteq \mathbb{A}^n$. Note that $\mathrm{Mor}(X, \mathbb{A}^n)$ has only a countable dimension as a $\mathbb{C}$-vector space. Thus, it has the structure of an ind-variety by the same construction as in Example 2.3 (3). It follows, that

$$\mathrm{Mor}(X, Y) = \{f \in \mathrm{Mor}(X, \mathbb{A}^n) \mid \varphi \circ f = 0 \text{ for all } \varphi \in \mathscr{I}(Y)\} = \bigcap_{\varphi \in \mathscr{I}(Y)} h_\varphi^{-1}(\{0\})$$

with $h_\varphi : \mathrm{Mor}(X, \mathbb{A}^n) \to \mathcal{O}(X), f \mapsto \varphi \circ f$ is closed in $\mathrm{Mor}(X, \mathbb{A}^n)$ and thus it has the structure of an ind-variety. □

We consider the following lemma about sets of morphisms which we use to show that $\mathrm{Aut}(X)$ has the structure of an ind-variety for an affine variety $X$.

**2.5 Lemma.** Let $X, Y$ and $Z$ be affine varieties. Then we have a bijection

$$\mathrm{Mor}(X \times Y, Z) \to \mathrm{Mor}(X, \mathrm{Mor}(Y, Z)), f \mapsto (x \mapsto (y \mapsto f(x, y))).$$

In fact, the bijection is an isomorphism of ind-varieties.

*Proof.* The given map is a bijection, because we have the inverse map

$$\mathrm{Mor}(X, \mathrm{Mor}(Y, Z)) \to \mathrm{Mor}(X \times Y, Z), g \mapsto ((x, y) \mapsto g(x)(y)).$$

The bijection is an isomorphism which follows by its structure. □

Now we are ready to prove the main statement of the current chapter.

**2.6 Theorem** (Stampfli)**.** Let $X$ be an affine variety. Then $\mathrm{Aut}(X)$ has the structure of an ind-group such that for any algebraic group $G$, the $G$-action on $X$ corresponds to the ind-group homomorphism $G \to \mathrm{Aut}(X)$.

*Proof.* [23, Proposition 3.7] Let $n \in \mathbb{N}$ such that $X \subseteq \mathbb{A}^n$. Next we consider the canonical $\mathbb{C}$-linear projection $p : \mathrm{End}(\mathbb{A}^n) \to \mathrm{Mor}(X, \mathbb{A}^n)$. Thus, a filtration of $\mathrm{Mor}(X, \mathbb{A}^n)$ is $p(\mathrm{End}(\mathbb{A}^n)_d)$ with the notation from Example 2.3. Set $\mathrm{End}(X)_i := \mathrm{End}(X) \cap p(\mathrm{End}(\mathbb{A}^n)_i)$. This means that $\mathrm{End}(X)$ is an ind-variety with the filtration $\mathrm{End}(X)_i$. From the construction we see that

$$\mathrm{End}(X) \times \mathrm{End}(X) \to \mathrm{End}(X), (f, g) \mapsto f \circ g$$

is a morphism because for any $f, g \in \mathrm{End}(X)_i$ there exists $m \in \mathbb{N}$ such that $f \circ g \in \mathrm{End}(X)_m$.

The set

$$\mathrm{Aut}(X) = \{(f, g) \in \mathrm{End}(X) \times \mathrm{End}(X) \mid f \circ g = g \circ f = \mathrm{id}\}$$

is closed in $\mathrm{End}(X) \times \mathrm{End}(X)$ and it has the structure of an ind-variety. Since the composition in $\mathrm{End}(X)$ is a morphism we consider

$$\mathrm{Aut}(X) \times \mathrm{Aut}(X) \to \mathrm{Aut}(X), ((f_1, h_1), (f_2, h_2)) \mapsto (f_1 \circ f_2, h_1 \circ h_2),$$

which is a morphism as well. The inverse map is

$$\mathrm{Aut}(X) \to \mathrm{Aut}(X), (f, h) \mapsto (h, f),$$

which is a morphism, too. Thus, $\mathrm{Aut}(X)$ is an ind-group.

Let $G$ be an algebraic group that acts on $X$ via a morphism $\rho : G \times X \to X$. Then the map $\rho_g : X \to X, x \mapsto \rho(g, x)$ is an endomorphism on $X$. By Lemma 2.5 the map $G \to \mathrm{End}(X), g \mapsto \rho_g$ is a morphism. Hence $G \to \mathrm{End}(X) \times \mathrm{End}(X), g \mapsto (\rho_g, \rho_g^{-1})$ is a morphism. We obtain a homomorphism of $G \to \mathrm{Aut}(X)$ by the construction of $\mathrm{Aut}(X)$ from $\mathrm{End}(X) \times \mathrm{End}(X)$.

Conversely, let $G \to \mathrm{Aut}(X)$ be a homomorphism of ind-groups. Then

$$G \to \mathrm{Aut}(X) \subseteq \mathrm{End}(X) \times \mathrm{End}(X) \to \mathrm{End}(X)$$

is a morphism, since $\mathrm{End}(X) \times \mathrm{End}(X) \to \mathrm{End}(X)$ is the projection onto the first coordinate. Thus $G \times X$ is a $G$-action by Lemma 2.5. □

In Theorem 2.6, we viewed elements from $\mathrm{Aut}(X)$ as

$$\{(f, g) \in \mathrm{End}(X) \times \mathrm{End}(X) \mid f \circ g = g \circ f = \mathrm{id}\}.$$

This is isomorphic to $\mathrm{Aut}(X) = \{f \in \mathrm{Mor}(X, X) \mid f : X \to X \text{ is an automorphism}\}$ by the projection onto the first coordinate.
Now we prove other properties of ind-groups which we use in the last chapter of this thesis.

**2.7 Lemma.** Let $G$ be an ind-group that acts on an ind-variety $X$. Then the stabilizer $\mathrm{Stab}_G(x)$ of $x \in X$ is a closed subset in $G$.

*Proof.* Let $\rho : G \times X \to X$ be the $G$-action on $X$. Then we define $\rho_x : G \to X$ given by $g \mapsto \rho(g, x)$. This map is a morphism because $\rho_x = pr_1(\rho_{|G \times \{x\}})$ with $pr_1$ the projection onto the first coordinate and $\rho_{|G \times \{x\}}$ the restricted of $\rho$ to $G \times \{x\}$. Now consider $\rho_x^{-1}(x) = \{g \in G \mid \rho(g, x) = x\} = \mathrm{Stab}_G(x)$. This set has to be closed, because it is the preimage of the closed subset $\{x\} \subseteq X$. □

**2.8 Corollary.** Let $H \subseteq G$ be an ind-subgroup of an ind-group $G$. Then the centralizer, i.e. the set $\{g \in G \mid \forall h \in H : ghg^{-1} = h\}$, is a closed subset in $G$.

*Proof.* Let the ind-group $G$ act on the ind-variety $G$ by conjugation. Then we observe that the stabilizer $\bigcap_{h \in H} \mathrm{Stab}_G(h)$ and the centralizer of $H$ coincide. By Lemma 2.7 the stabilizer $\mathrm{Stab}_G(h)$ is closed and thus the centralizer has to be closed as well. $\square$

**2.9 Lemma.** Let $X$ and $Y$ be affine varieties, let $\varphi : \mathrm{Aut}(X) \to \mathrm{Aut}(Y)$ an isomorphism of ind-groups and $G$ an algebraic subgroup of $\mathrm{Aut}(X)$. Then $\varphi(G)$ is an algebraic subgroup of $\mathrm{Aut}(Y)$.

*Proof.* We have to show that $\varphi(G)$ is closed. The subgroup $\varphi(G) \subseteq \mathrm{Aut}(Y)$ is the preimage of $G \subseteq \mathrm{Aut}(X)$ under the isomorphism $\varphi^{-1} : \mathrm{Aut}(Y) \to \mathrm{Aut}(X)$. Thus, $\varphi(G) \subseteq \mathrm{Aut}(Y)$ is the preimage of a closed subset of $\mathrm{Aut}(X)$ and in particular is closed itself. $\square$

For an ind-group or algebraic group $G$ we denote by $G^\circ$ the connected component of the neutral element, i.e. the connected component of $G$ that contains the neutral element. Lemma 1.3 shows that $G^\circ$ is closed in $G$ and thus is an ind-subgroup, or algebraic subgroup respectively.

**2.10 Lemma.** Let $\varphi : \mathrm{Aut}(X) \to \mathrm{Aut}(Y)$ an isomorphism of ind-groups, then $\varphi$ induces the isomporphism from $\mathrm{Aut}^\circ(X)$ to $\mathrm{Aut}^\circ(Y)$.

*Proof.* First, an isomporphism sends the neutral element to the neutral element. Lemma 1.4 further shows that the image of a connected subset is again connected under the isomorphism $\varphi$, since morphisms are continuous maps. Analogously we obtain that the preimage of a connected subset is connected under $\varphi$ because the inverse map of an isomorphism is still continuous. Thus, we have $\varphi^{-1}(\mathrm{Aut}^\circ(Y)) \subseteq \mathrm{Aut}^\circ(X)$ and similarly $\varphi(\mathrm{Aut}^\circ(X)) \subseteq \mathrm{Aut}^\circ(Y)$. Hence, $\varphi$ restricted to $\mathrm{Aut}^\circ(X)$ is an isomporphism between $\mathrm{Aut}^\circ(X)$ and $\mathrm{Aut}^\circ(Y)$. $\square$

**2.11 Lemma.** Let $G$ be an ind-group. Then $G^\circ \subseteq G$ is a normal subgroup of countable index.

*Proof.* This proven in [4, Proposition 2.2.1]. $\square$

If we replace $D_p$ by $\mathbb{A}^n$ in the Main Theorem A, we obtain a stronger result.

**2.12 Theorem.** Let $X$ be a connected affine variety. If $\mathrm{Aut}(X) \cong \mathrm{Aut}(\mathbb{A}^n)$ as an abstract group, then $X \cong \mathbb{A}^n$ as a variety.

*Proof.* This theorem is proven in [17, Theorem A]. $\square$

In the next chapter we cite properties that only have been proven for Danielewski surfaces $D_p$ where $\deg p \geq 3$. Therefore, it is necessary to consider the cases, where $\deg p \leq 2$.

**2.13 Lemma.** Let $p$ be a polynomial with simple roots. We then have

$$D_p \cong \begin{cases} \mathbb{A}^2, & \text{if } \deg p = 1, \\ \mathrm{SL}_2/T, & \text{if } \deg p = 2, \end{cases}$$

where $T$ denotes the torus $\left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mid t \in \mathbb{C}^* \right\}$.

*Proof.* First we consider the case $\deg p = 1$. Choose $a, b \in \mathbb{C}$ such that $p(z) = az+b$. Thus, we consider the map

$$\mathbb{A}^2 \to D_p, (x,y) \mapsto \left( x, y, \frac{xy-b}{a} \right).$$

This map is well-defined because $a$ is not 0 as $\deg p = 1$. This map gives an isomorphism with the following calculation

$$(x,y,z) \in D_p \Leftrightarrow xy = az+b \Leftrightarrow xy - b = az \Leftrightarrow \frac{xy-b}{a} = z.$$

Then we consider the case of $\deg p = 2$. Our strategy is to show that $D_p$ is isomorphic to $D_{z^2-1}$ and then apply Lemma 1.36. Choose $a, b, c \in \mathbb{C}$ such that $p(z) = a(z-b)^2 - c$. Note, that $a$ and $c$ cannot be 0, because $p$ has simple roots and is of degree 2. Consider the map

$$\delta : D_p \to D_{z^2-1}, (x,y,z) \mapsto \left( cx, y, z \cdot \sqrt{\frac{c}{a}} + b \right).$$

This map is an isomorphism, because we have

$$\begin{aligned}
(x',y',z') \in D_p \Leftrightarrow x'y' = a(z'-b)^2 - c &\overset{\delta}{\Leftrightarrow} cxy = a\left( z \cdot \sqrt{\frac{c}{a}} + b - b \right)^2 - c \\
&\Leftrightarrow cxy = a\left( z \cdot \sqrt{\frac{c}{a}} \right)^2 - c \Leftrightarrow cxy = \frac{ac}{a}(z)^2 - c \\
&\Leftrightarrow cxy = cz^2 - c \Leftrightarrow xy = z^2 - 1. \\
&\Leftrightarrow (x,y,z) \in D_{z^2-1}
\end{aligned}$$

Thus, all Danielewski surfaces $D_p$ with a polynomial $p$ of degree 2 and simple roots are isomorphic to each other. By Lemma 1.36 the proof follows. □

Lemma 2.13 combined with Theorem 2.12 gives us the statement of the main results in the case of Danielewski surfaces with polynomials of degree 1. To obtain our claim for polynomials of degree 2, we need the following result.

**2.14 Proposition.** Let $X$ be an irreducible variety and $\mathrm{Aut}(X) \cong \mathrm{Aut}(\mathrm{SL}_2/T)$ as an ind-group. Then $X \cong \mathrm{SL}_2/T$ as a variety.

*Proof.* This is proven in [16, Theorem 1.4]. □

# 3 Root subgroups of Aut$(D_p)$

In this chapter we concentrate on the automorphism group of a Danielewski surface. From now on we assume $p$ to be a polynomial over $\mathbb{C}$ with degree at least 3 and with only simple roots. This restriction of the degree is justified by the results from the previous chapter, where we already considered the cases $\deg p = 1, 2$. To describe the structure of $\mathrm{Aut}(D_p)$ we first need to define the semidirect product and the free product.

**3.1 Definition.** Let $N, H$ be groups and $\psi : H \to \mathrm{Aut}(N), h \mapsto \psi_h(n) = hnh^{-1}$ a homomorphism. Then we define the group $G' = (N, H)$ with the following group operation for $(n_1, h_1), (n_2, h_2) \in G'$:

$$(n_1, h_1)(n_2, h_2) = (n_1\psi_{h_1}(n_2), h_1h_2) = (n_1h_1n_2h_1^{-1}, h_1h_2).$$

This group $G'$ is called the *semidirect product* and we denote it with $N \rtimes H$.

**3.2 Definition.** Let $G$ and $H$ be groups. A *word* is a string of elements from $G$ and $H$. Such a word can be *reduced* by

- removing an instance of the identity element,
- replacing a pair $g_1g_2$ by its product in $G$, or a pair $h_1h_2$ by its product in $H$.

If a word cannot be reduced anymore, it is in its *reduced form* and we call it *irreducible*. The *free group* is the set of all irreducible finite words where the group operation is concatenation followed by maximal reduction. We denote the free group with $H * G$.

**3.3 Remark.** The elements of $G * H$ are uniquely determent by the elements in their reduced form. In other words two elements, of $G * H$ are different, if they have different elements in their reduced form. We obtain this by

$$u_1v_1 \dots u_nv_n = u'_1v'_1 \dots u'_kv'_k \Leftrightarrow u_1v_1 \dots u_nv_nv'^{-1}_k u^{-1}_k \dots v'^{-1}_1 u'^{-1}_1 = \mathrm{id}\,.$$

Note that the identity is only possible, if $n = k, v_i = v'_i$ and $u_i = u'_i$.

The structure of the automorphism group of $D_p$ is described in [12]. In this thesis we follow the notations of [9, Section 5] to describe the structure of $\mathrm{Aut}(D_p)$ and $\mathrm{Aut}(D_p)^\circ$.

$$\begin{aligned}
T &:= \{(x,y,z) \mapsto (cx, c^{-1}y, z) \mid c \in \mathbb{C}^*\} \cong \mathbb{C}^*,\\
U_x &:= \{(x,y,z) \mapsto (x, x^{-1}p(z+g(x)), z+g(x)) \mid g \in x\mathbb{C}[x]^+\},\\
U_y &:= \{(x,y,z) \mapsto (y^{-1}p(z+h(y)), y, z+h(y)) \mid h \in y\mathbb{C}[y]^+\}.
\end{aligned}$$

**3.4 Remark.** We note that the groups $U_x$ and $U_y$ do not commute. Indeed, let $u = (x, x^{-1}p(z+g(x)), p(z+g(x)) \in U_x), v = (y^{-1}p(z+h(y)), y, z+h(y)) \in U_y$ be two elements. Then $u \circ v$ and $v \circ u$ are given by

$$u \circ v : (x,y,z) \mapsto$$
$$\left(y^{-1}p(z+h(y)), \frac{p(z+h(y)+g(y^{-1}p(z+h(y))))}{y^{-1}p(z+h(y))}, z+h(y)+g(y^{-1}p(z+h(y)))\right)$$

and

$$v \circ u : (x,y,z) \mapsto$$
$$\left(\frac{p(z+g(x)+h(x^{-1}p(z+g(x))))}{x^{-1}p(z+g(x))}, x^{-1}p(z+g(x)), z+g(x)+h(x^{-1}p(z+g(x)))\right).$$

This shows that $v \in U_y$ and $u \in U_x$ do not commute.

Define $I$ as the subgroup of $\mathrm{Aut}(D_p)$ generated by the involution $\tau : (x,y,z) \mapsto (y,x,z)$.
Let $\Gamma$ be the subgroup generated by $\gamma : (x,y,z) \mapsto (x, \mu y, az+b)$, where $\mu, a, b \in \mathbb{C}$ such that $\mu p(z) = p(az+b)$. If $p$ has simple roots, $\Gamma$ is finite. The set $\Gamma$ can be trivial which is the case if there does not exist an automorphism of $\mathbb{C}$ permuting the roots of $p$. By [9, Proposition 5] we obtain

$$\mathrm{Aut}(D_p) = (U_x * U_y) \rtimes ((T \times \Gamma) \rtimes I), \tag{2}$$
$$\mathrm{Aut}^\circ(D_p) = (U_x * U_y) \rtimes T. \tag{3}$$

As we know from the preliminaries, $\mathcal{O}(D_p)$ can be written as $\mathbb{C}[x,y,z]/(xy - p(z))$. We observe that multiples of $xy$ can be written as multiples of $p(z)$. Thus, $\mathcal{O}(D_p)$ cannot contain multiples of $xy$, if we use $\mathbb{C}[z]$. This illustrates the description

$$\mathcal{O}(D_p) = x\mathbb{C}[x,z] \oplus y\mathbb{C}[y,z] \oplus \mathbb{C}[z].$$

**3.5 Remark.** It is important to note that $U_x$ and $U_y$ are normalized by $T$.
Let $u = (x, x^{-1}p(z+g(x)), z+g(x)) \in U_x$ and $t = (cx, c^{-1}y, z)$. Thus, we derive

$$\begin{aligned} &t \circ u \circ t^{-1} : \\ &(x,y,z) \mapsto (cx, c^{-1}y, z) \circ (x, x^{-1}p(z+g(x)), z+g(x)) \circ (c^{-1}x, cy, z) \\ &\qquad = (cx, c^{-1}y, z) \circ (c^{-1}x, (c^{-1}x)^{-1}p(z+g(c^{-1}x)), z+g(c^{-1}x)) \\ &\qquad = (cc^{-1}x, c^{-1}(c^{-1}x)^{-1}p(z+g(c^{-1}x)), z+g(c^{-1}x)) \\ &\qquad = (x, x^{-1}p(z+g(c^{-1}x)), z+g(c^{-1}x)) \end{aligned}$$

and analogously we get

$$t \circ v \circ t^{-1} : (x,y,z) \mapsto (y^{-1}p(z+h(cy)), y, z+h(cy))$$

for $v = (y^{-1}p(z+h(y)), y, z+h(y)) \in U_y$.

This calculation also shows that no non-trivial element from $T$ commutes with a non-trivial element from $U_x$ nor $U_y$.

**3.6 Lemma.** The centralizer of $T \subseteq \mathrm{Aut}(D_p)$ is $T \times \Gamma$.

*Proof.* First, we consider the composition of $I$ and $T$. Let $t = (cx, c^{-1}y, z) \in T$ and $\tau = (y,x,z) \in I$. Then

$$\begin{aligned} &\tau \circ t \circ \tau^{-1} : \\ &(x,y,z) \mapsto (y,x,z) \circ (cx, c^{-1}y, z) \circ (y,x,z) \\ &\qquad = (y,x,z) \circ (cy, c^{-1}x, z) \\ &\qquad = (c^{-1}x, cy, z). \end{aligned}$$

Thus, $I$ normalizes $T$, but does not commute with it.

The next step is to consider the composition of $T$ with $U_x * U_y$. From Remark 3.5 we already know that the centralizer of $T$ in $\mathrm{Aut}(D_p)$ does not contain any non-trivial elements from both $U_x$ or $U_y$. We check now that a non-trivial element from $T$ does not commute with a non-trivial element from $U_x * U_y$.

Let $t \in T$ and $u_1 v_1 \dots u_n v_n \in U_x * U_y$. We consider

$$tu_1v_1 \dots u_nv_nt^{-1} = tu_1t^{-1}tv_1t^{-1}t \dots t^{-1}tu_nt^{-1}tv_nt^{-1} = u'_1v'_1 \dots u'_nv'_n$$

with $u'_i := tu_it^{-1}$ and $v'_i := tv_it^{-1}$ respectively. From Remark 3.5 we know that $u_i$ and $u'_i$ are different. By Remark 3.3 we have $u_1v_1 \dots u_nv_n \neq u'_1v'_1 \dots u'_nv'_n$. This shows that $u_1v_1 \dots u_nv_n$ is not an element of the centralizer of $T$.

Analogously, we obtain that $T$ does not commute with any element that is the product of $x \in U_x * U_y$ and $\tau \in I \backslash \{\mathrm{id}\} = \{\tau\}$. This follows because $x\tau \in (U_x * U_y) \rtimes I$,

$$tx\tau t^{-1} = txt^{-1}t\tau t^{-1} = (txt^{-1})(\tau t^{-2})$$

where $txt^{-1} \in U_x * U_y$ and $txt^{-1}\tau t^{-2} \notin (U_x * U_y) \rtimes I$.

Finally we show that $t = (cx, c^{-1}y, z) \in T$ commutes with $\gamma = (x, \mu y, az + b) \in \Gamma$, where $\mu p(z) = p(az + b)$. Indeed,

$$\begin{aligned}
&\gamma \circ t \circ \gamma^{-1} : \\
&(x, y, z) \mapsto (x, \mu y, az + b) \circ (cx, c^{-1}y, z) \circ (x, \mu^{-1}y, \frac{z - b}{a}) \\
&\qquad = (x, \mu y, az + b) \circ (cx, c^{-1}\mu^{-1}y, \frac{z - b}{a}) \\
&\qquad = (cx, c^{-1}y, z)
\end{aligned}$$

and in particular $\gamma \circ t = t \circ \gamma$. □

**3.7 Remark.** Now we consider the structure of $U_x$ and $U_y$. Let

$$\varphi : x\mathbb{C}[X]^+ \to U_x, g(x) \mapsto (x, x^{-1}p(z + g(x)), z + g(x)).$$

We claim that $\varphi$ is the isomorphism. Let $u_1 = \varphi(g_1) = (x, x^{-1}p(z + g_1(x)), z + g_1(x)) \in U_x, u_2 = \varphi(g_2) = (x, x^{-1}p(z + g_2(x)), z + g_2(x)) \in U_x$ be two elements. Then we consider the composition

$$\begin{aligned}
u_1 \circ u_2 = \varphi(g_2) \circ \varphi(g_2) :& \\
(x, y, z) \mapsto& (x, x^{-1}p(z + g_1(x)), z + g_1(x)) \circ (x, x^{-1}p(z + g_2(x)), z + g_2(x)) \\
=& (x, x^{-1}p(z + g_1 + g_2(x)), z + g_1 + g_2(x)) \\
=& (x, x^{-1}p(z + (g_1 + g_2)(x)), z + (g_1 + g_2)(x)).
\end{aligned}$$

This shows that $\varphi(g_1 + g_2) = \varphi(g_1) + \varphi(g_2)$. Thus, $U_x$ is isomorphic to $x\mathbb{C}[x]^+$ as a group. Furthermore, $U_x$ is an ind-group. A filtration is given by

$$(U_x)_d := \{(x, y, z) \mapsto (x, x^{-1}p(z + g(x)), z + g(x) \mid g \in x\mathbb{C}[x]^+, \deg g \leq d\}.$$

and by Example 2.3 $\varphi$ is the isomorphism of ind-groups. Analogously we obtain that the map

$$y\mathbb{C}[y]^{+} \to U_y, h \mapsto (y^{-1}p(z+h(y)), y, z+h(y))$$

is the isomorphism of ind-groups.

**3.8 Lemma.** The centralizer of $U_x \subseteq \mathrm{Aut}(D_p)$ and of $U_y \subseteq \mathrm{Aut}(D_p)$ coincides with $U_x$ and $U_y$ respectively.

*Proof.* The cases of $U_x$ and $U_y$ are analogous to each other. Therefore, we only have to consider $U_x$.

Now we check if $U_x$ commutes with any of the other subgroups $T, I, \Gamma, U_x * U_y$. In Lemma 3.6 it is proved that groups $T$ and $U_x$ do not commute. Now, let $u = (x, x^{-1}p(z+g(x)), z+g(x)) \in U_x$ and $\tau = (y, x, z) \in I$. Then

$$\begin{aligned}
&\tau \circ u \circ \tau^{-1}: \\
&(x,y,z) \mapsto (y,x,z) \circ (x, x^{-1}p(z+g(x)), z+g(x)) \circ (y,x,z) \\
&\qquad\qquad = (y,x,z) \circ (y, y^{-1}p(z+g(y)), z+g(y))) \\
&\qquad\qquad = (y^{-1}p(z+g(y)), y, z+g(y))
\end{aligned}$$

and thus $\tau$ is not an element of the centralizer of $U_x$.

Next we take $\gamma = (x, \mu y, az+b) \in \Gamma$ with $\mu p(z) = p(az+b)$. Then

$$\begin{aligned}
&\gamma \circ u \circ \gamma^{-1}: \\
&(x,y,z) \mapsto (x, \mu y, az+b) \circ \left(x, x^{-1}p\left(z+g\left(x\right)\right), z+g\left(x\right)\right) \circ \left(x, \mu^{-1}y, \frac{z-b}{a}\right) \\
&\qquad = (x, \mu y, az+b) \circ \left(x, x^{-1}p\left(\frac{z-b}{a}+g\left(x\right)\right), \frac{z-b}{a}+g\left(x\right)\right) \\
&\qquad = \left(x, \mu x^{-1}p\left(\frac{z-b}{a}+g\left(x\right)\right), z-b+ag\left(x\right)+b\right) \\
&\qquad = \left(x, x^{-1}p\left(z+ag\left(x\right)\right), z+ag\left(x\right)\right).
\end{aligned}$$

The last equality follows because

$$\mu p\left(\frac{z-b}{a}+g\left(x\right)\right) = p\left(a\left(\frac{z-b}{a}+g\left(x\right)\right)+b\right) = p\left(z+ag\left(x\right)\right).$$

Thus, non-trivial elements from $U_x$ and $\gamma$ do not commute.

Finally, Remark 3.4 and Remark 3.7 show that the only elements from $U_x * U_y$ that commute with each element of $U_x$ are the elements from $U_x$ itself. Let $u \in U_x, t \in T, \nu \in I, \gamma \in \Gamma$ and $z \in U_x * U_y$. Assume that $zt\gamma\nu$ commutes with each element $u \in U_x$. If $z \notin U_x$ then we claim that

$$u = (zt\gamma\nu)^{-1}u(zt\gamma\nu) = (\nu^{-1}\gamma^{-1}t^{-1}z^{-1})u(zt\gamma\nu) \in U_x * U_y \setminus U_x \tag{4}$$

which contradicts the assumption that $u \in U_x$. (4) follows because $I, T, \Gamma$ normalise $U_x \cup U_y$ and hence they do not change the number of elements in the word

$z^{-1}uz \in U_x * U_y \setminus U_x$. Thus, $z \in U_x$ and $zt\gamma\nu$ commutes with $U_x$ if and only if $t\gamma\nu$ commutes with $U_x$. Assume next that $\nu \neq \mathrm{id}$. Then $\nu u \nu^{-1} \in U_y$ and in particular $t\gamma\nu u\nu^{-1}\gamma^{-1}t^{-1} \in U_y$ because $T$ and $\Gamma$ normalise $U_y$. Thus, $\nu = \mathrm{id}$. Suppose $\gamma = (x, \mu y, az + b) \in \Gamma, t = (cx, c^{-1}y, z) \in T$ and $u = (x, x^{-1}p(z + g(x)), z + g(x)) \in U_x$ then we obtain the following:

$$t\gamma u\gamma^{-1}t^{-1} = (x, x^{-1}p(z + ag(c^{-1}x)), z + ag(c^{-1}x))$$

Thus, the product $t\gamma$ does not lie in the centralizer of $U_x$ and the proof follows. □

**3.9 Remark.** We already saw that $T$ normalizes $U_x$ and $U_y$. The proofs of Lemma 3.6 and 3.4 show further interactions between the subgroups. The involution $\tau = (y, x, z) \in I$ normalizes $t \in T$ with $\tau \circ t \circ \tau = t^{-1}$ and fulfills $U_x = \tau U_y \tau^{-1}$. Furthermore, $\Gamma$ normalizes $U_x, U_y$ and commutes with $T$. Since $U_x * U_y$ is a string of elements of $U_x$ and $U_y$, it is normalized by $T$, $I$ and $\Gamma$ as well.

In the following we focus on special subgroups because they are important for the proof of the main results.

**3.10 Definition.** Let $X$ be an affine variety. A subgroup $U \subseteq \mathrm{Aut}(X)$ is called a *root subgroup* with respect to $T' \cong (\mathbb{C}^*)^k \subseteq \mathrm{Aut}(X)$ if $U$ is an algebraic subgroup isomorphic to $\mathbb{C}^+$ and normalized by $T'$.

To each root subgroup we attach a so-called weight. To define the weight, we must consider other lemmas.

**3.11 Lemma.** Let $\varphi$ be a homomorphism of algebraic groups from $\mathbb{C}^*$ to $\mathbb{C}^*$. Then there exists $r \in \mathbb{Z}$ such that for all $x \in \mathbb{C} \setminus \{0\}$ we have $\varphi(x) = x^r$.

*Proof.* The described maps are obviously all endomorphisms of $\mathbb{C}^*$. Therefore, the task is to show that there is no other endomorphism.

Let $\varphi$ be an endomorphism of $\mathbb{C}^*$. Consider the pull back $\varphi^* : \mathcal{O}(\mathbb{C}^*) \to \mathcal{O}(\mathbb{C}^*)$ which we defined in Remark 1.15. By Lemma 1.17 we know that $\mathcal{O}(\mathbb{C}^*)$ and $\mathbb{C}[t, t^{-1}]$ are isomorphic. With that we have

$$\varphi^* : \mathbb{C}[t, t^{-1}] \to \mathbb{C}[t, t^{-1}], \ f(t) \mapsto f(\varphi(t)).$$

Computing $\varphi^*(t)$ with $t \in \mathbb{C}[t, t^{-1}]$ gives us the original $\varphi$ as the image. In particular, we know that $\varphi$ is an element of $\mathbb{C}[t, t^{-1}]$. Thus, there exist $g, h \in \mathbb{C}[t]$ with no common root, such that $\varphi = g/h$. Next, we consider the properties of $\varphi$. We note that $\varphi$ is an endomorphism of $\mathbb{C}^*$. We see that $h$ has no roots other than 0, since $\varphi$ would not be well-defined at those roots. This argument also holds for roots of $g$ because $g(x)$ belongs to $\mathbb{C}^*$ for any $x \in \mathbb{C}^*$. With those two arguments $\varphi$ has to have the form $ct^k$ for $c \in \mathbb{C}^*$ and $k \in \mathbb{Z}$.

If we assume that $c$ is not equal to 1, then we get

$$\varphi(1) \cdot \varphi(1) = c^2 \neq c = \varphi(1 \cdot 1).$$

This contradicts the fact that $\varphi$ is a homomorphism. Therefore, $c$ must be equal to 1 which concludes the proof. □

From Lemma 3.11 we further obtain a classification of all automorphisms of $\mathbb{C}^*$.

**3.12 Corollary.** Let $\varphi$ be an automorphism of algebraic groups from $\mathbb{C}^*$ to $\mathbb{C}^*$. Then there exists $r \in \{-1, 1\}$, such that for all $x \in \mathbb{C} \setminus \{0\}$ we have $\varphi(x) = x^r$.

*Proof.* By Lemma 3.11 we already have that $\varphi(x) = x^r$ for some $r \in \mathbb{Z}$. Thus, we only need to prove that $r \in \mathbb{Z} \setminus \{-1, 1\}$ is not possible. The kernel of $\varphi$ is trivial. If $r \in \mathbb{Z} \setminus \{-1, 1\}$, we always have the roots of unity as non-trivial kernel. Thus, $\varphi(x) = x^r$ is an automorphism if and only if $r \in \{-1, 1\}$. □

**3.13 Lemma.** Let $A$ be the set of $\mathbb{C}^*$-actions on $\mathbb{C}^+$. Then there exists a bijection $w : A \to \mathbb{Z}$ such that $\rho(g, x) = g^{w(\rho)}x$ for all $\rho \in A, g \in \mathbb{C}^*$ and $x \in \mathbb{C}^+$.

*Proof.* Let $\rho : \mathbb{C}^* \times \mathbb{C}^+ \to \mathbb{C}^+$ be a $\mathbb{C}^*$-action on $\mathbb{C}^+$. We define

$$\rho_g : \mathbb{C}^+ \to \mathbb{C}^+, x \mapsto \rho(g, x),$$

for $g \in \mathbb{C}^*$. Since $\rho_g$ has the inverse map $\rho_{g^{-1}}$, it is from $\mathrm{Aut}(\mathbb{C}^+)$. By Lemma 1.31 we have $\mathrm{Aut}(\mathbb{C}^+) \cong \mathbb{C}^*$ and that there exists $c_g \in \mathbb{C}^*$, such that $\rho_g(x) = c_g x$.

As the next step, we consider the map

$$\rho' : \mathbb{C}^* \to \mathbb{C}^*, h \mapsto c_h.$$

Since $\rho_g \circ \rho_h = \rho_{gh}$ for all $g, h \in \mathbb{C}^*$, $\rho'$ has to be an homomorphism. By Lemma 3.11 there exists $r \in \mathbb{Z}$ such that $\rho'(h) = h^r$. Thus $\rho(g, x) = \rho_g(x) = c_g x = g^r x$ which gives us the map $w : A \to \mathbb{Z}, \rho \mapsto r$. This map is obviously injective. Moreover, $w$ is surjective since for any $r \in \mathbb{Z}$ $\rho(g, x) = g^r x$ is a $\mathbb{C}^*$-action of $\mathbb{C}^+$. □

**3.14 Remark.** (1) We call $r = w(\rho) \in \mathbb{Z}$ from Lemma 3.13 the *weight* of a $\mathbb{C}^*$-action $\rho$.
(2) Note that we can apply an automorphism of $\mathbb{C}^*$, to the $\mathbb{C}^*$-action. By Corollary 3.12 the only non-trivial automorphism of $\mathbb{C}^*$ is given by the map $g \mapsto g^{-1}$. This implies that it is possible to change the sign of $r \in \mathbb{Z}$. Hence, depending on the action of a torus, the weight of $\mathbb{C}^*$-action on $\mathbb{C}^+$ is $-n$ or $n$ with $n \in \mathbb{N}$. But the absolute value cannot change.
(3) It is important to see that this change of the sign can only effect all weights together. Thus, if two root subgroups have different weights with respect to one action of the torus, they have different weights with respect to another torus.
(4) From now on we consider the weight up to torus automorphisms which means that the sign can change. But this is not a problem, since we concentrate on cases, where the signs are different.

**3.15 Corollary.** Let $\rho$ be a non-trivial $\mathbb{C}^*$-action on $\mathbb{C}^+$. Then $\rho$ acts transitively on $\mathbb{C} \setminus \{\mathrm{id}\}$.

*Proof.* By Lemma 3.13 there exists $r \in \mathbb{Z}$, with $\rho : \mathbb{C}^* \times \mathbb{C}^+ \to \mathbb{C}^+, (g, x) \mapsto g^r x$. Since $\rho$ is non-trivial, $r$ cannot be 0.
Thus, the action has to obviously be transitive on $\mathbb{C}^+\{0\}$. □

The next goal is to define root subgroups of $\mathrm{Aut}(D_p)$ with respect to $T$, and consider their weights. Let $n$ be an element from $\mathbb{N}_1$. We then define

$$U_{x,n} := \{(x, y, z) \mapsto (x, x^{-1}p(z + sx^n), z + sx^n) \mid s \in \mathbb{C}\},$$

$$U_{y,n} := \{(x, y, z) \mapsto (y^{-1}p(z + ry^n), y, z + ry^n) \mid r \in \mathbb{C}\}.$$

**3.16 Lemma.** Let $n \in \mathbb{N}_1$, then $U_{x,n}$ and $U_{y,n}$ are root subgroups with respect to $T$. Moreover, $U_{x,n}$ has the weight $-n$, and $U_{y,n}$ has the weight $n$ up to a torus automorphism.

*Proof.* First we have to verify that both $U_{x,n}$ and $U_{y,n}$ are isomorphic to $\mathbb{C}^+$. Let $u = (x, x^{-1}p(z+sx^n), z+sx^n)$ and $v = (x, x^{-1}p(z+rx^n), z+rx^n)$ be elements from $U_{x,n}$, with $s, r \in \mathbb{C}$. Then Remark 3.7 shows that

$$u \circ v = (x, x^{-1}p(z + (r + s)x^n), z + (r + s)x^n).$$

Thus, the composition in $U_{x,n}$ is well-defined and the isomorphism $\varphi : \mathbb{C} \to U_{x,n}$ is given by

$$g \mapsto (x, x^{-1}p(z + gx^n), z + gx^n).$$

The same claim follows analogously for $U_{y,n}$.

Now we need to check whether $T$ normalizes $U_{x,n}$. Let $u = (x, x^{-1}p(z + sx^n), z + sx^n) \in U_{x,n}$ and $t = (cx, c^{-1}y, z) \in T$, where $s \in \mathbb{C}$ and $c \in \mathbb{C}^*$. With the calculation from Remark 3.5 we derive

$$t \circ u \circ t^{-1} : (x, y, z) \mapsto (x, x^{-1}p(z + sc^{-n}x^n), z + sc^{-n}x^n).$$

It follows, that $T$ normalizes $U_{x,n}$. Since $\varphi^{-1}(u) = s$ and $\varphi^{-1}(t \circ u \circ t^{-1}) = c^{-n}s$, the weight has to be $-n$ by the definition and Lemma 3.13. Analogously we get

$$t \circ v \circ t^{-1} : (x, y, z) \mapsto (y^{-1}p(z + sc^n y^n), y, z + sc^n y^n)$$

for $v = (y^{-1}p(z + sy^n), y, z + sy^n) \in U_{y,n}$. This shows that $U_{y,n}$ has the weight $n$. This concludes the proof. $\square$

Additionally, we consider another characterization for $U_{x,n}$.

**3.17 Lemma.** Let $n \in \mathbb{N}$ be and $t = (\sqrt[n]{2}x, (\sqrt[n]{2})^{-1}y, z) \in T$. Then

$$U_{x,n} = \left\{u \in U_x \mid t^{-1} \circ u \circ t = u^2\right\}.$$

*Proof.* We use the same notations as in Remark 3.5. Thus, we have

$$t^{-1} \circ u \circ t : (x, y, z) \mapsto (x, x^{-1}p(z + g(cx)), z + g(cx)),$$

where $t \in T$ and $u \in U_x$ with $c = \sqrt[n]{2} \in \mathbb{C}^*$ and $g(x) \in x\mathbb{C}[x]^+$ respectively. To satisfy

$$t^{-1} \circ u \circ t = u^2 = (x, x^{-1}p(z + 2g(x)), z + 2g(x)),$$

$g(x)$ must be a monomial of degree $n$. The proof follows. $\square$

In the following, we consider the relationship between the weights and the root subgroups in $\mathrm{Aut}(D_p)$. This is necessary for the proofs of the main results. We show that $U_{x,n}$ and $U_{y,n}$ are all root subgroups in $\mathrm{Aut}(D_p)$. Moreover, it is possible to determine root subgroups uniquely by their weights. Before we do that, we first introduce some notions.

If there exists a faithful regular $\mathbb{C}^*$-action on $X$, we consider the fixed points associated with this action. It is proven that there are only three types of those actions. There is *elliptic* $\mathbb{C}^*$-action, where there is only one attractive fixed point. A $\mathbb{C}^*$-action on $X$ with an infinite number of fixed points is *parabolic* $\mathbb{C}^*$-action. If there are finitely many non-attractive $\mathbb{C}^*$-fixed points in $X$, a $\mathbb{C}^*$-action is called *hyperbolic*.

**3.18 Lemma.** Let $S$ be a non-toric $\mathbb{C}^*$-surface. Then

(1) The surface $S$ admits root subgroups of different weights, if and only if $S$ is hyperbolic.

(2) If $S$ is hyperbolic, then all root subgroups have different weights.

*Proof.* This lemma is proven in [10, Lemma 4.16]. □

**3.19 Theorem.** The root subgroups $U_{x,n}, U_{y,n}$ are all root subgroups in $\mathrm{Aut}(D_p)$ with respect to $T$.

*Proof.* By Lemma 3.16 we know that $U_{x,n}$ and $U_{y,n}$ are root subgroups of $\mathrm{Aut}(D_p)$ with respect to $T$ of all possible weights except 0. Remark 3.6 also shows that we cannot have a root subgroup with the weight 0, since all of its elements have to commute with $T$. Thus, we have to show that all root subgroups of $\mathrm{Aut}(D_p)$ with respect to $T$ have different weights.

To prove that all root subgroups have different weights we use Lemma 3.18. Thus, we first check whether $D_p$ is non-toric. From the description of $\mathrm{Aut}(D_p)$ (see (2)) we know that there is no copy of $(\mathbb{C}^*)^2$ in $\mathrm{Aut}(D_p)$. Hence, $D_p$ is non-toric.

Since $D_p$ admits root subgroups $U_{x,1}, U_{x,2} \subseteq \mathrm{Aut}(D_p)$ with respect to $T$ of different weights, $D_p$ is hyperbolic $\mathbb{C}^*$-surface (Lemma 3.18 (1)). Hence, Lemma 3.18 (2) implies that all root subgroups of $\mathrm{Aut}(D_p)$ with respect to $T$ have different weights.

□

**3.20 Remark.** The most important implication of Theorem 3.19 is the fact that every root subgroups in $\mathrm{Aut}(D_p)$ is uniquely determined by its weight. We resume this in the next chapter.

# 4 Proof of main results

From now on we assume $X$ to be an affine variety which is also normal and irreducible. Moreover, we assume that $\varphi : \mathrm{Aut}(D_p) \to \mathrm{Aut}(X)$ is an isomorphism of abstract groups.

The following proposition is used in the prove of Lemma 4.2 where we show that $\varphi(T)$ and $\varphi(U_{x,1})$ are one-dimensional subgroups of $\mathrm{Aut}(X)$.

**4.1 Proposition.** Let $X$ be an affine variety and let $G, H \subseteq \mathrm{Aut}(X)$ be non-trivial commutative connected closed subgroups such that

(1) all non-trivial elementsan in $H$ are of infinite order,

(2) $G$ normalizes $H$,

(3) for all $h \in H \setminus \{\mathrm{id}_X\}$, the group $\{g \in G \mid g \circ h \circ g^{-1} = h\}$ is finite.

Then there exist increasing filtration by closed commutative algebraic subgroups $G_1 \subseteq G_2 \subseteq \ldots \subseteq G$ and $H_1 \subseteq H_2 \subseteq \ldots \subseteq H$, such that $G_k$ normalizes $H_k$ for all $k \geq 1$,

$$G = \bigcup_{k=1}^{\infty} G_k \text{ and } H = \bigcup_{k=1}^{\infty} H_k.$$

Moreover, for all $k \geq 1$ there exist $s_k \geq 0, r_k \geq 0$, such that

$$G_k \cong (\mathbb{C}^*)^{s_k} \text{ and } H_k \cong (\mathbb{C}^+)^{r_k},$$

and $s_k, r_k \geq 1$ for $k$ big enough.

*Proof.* This is proven in [18, Corollary 3.3]. □

**4.2 Lemma.** The subgroup $\varphi(T) \subseteq \mathrm{Aut}(X)$ is a one-dimensional torus and the subgroup $\varphi(U_{x,1}) \subseteq \mathrm{Aut}(X)$ is a root subgroup with respect to $\varphi(T)$.

*Proof.* We apply Proposition 4.1 to the subgroups $\overline{\varphi(T)}^\circ$ and $\overline{\varphi(U_{x,1})}^\circ$. Thus, we check, if $\overline{\varphi(T)}^\circ$ and $\overline{\varphi(U_{x,1})}^\circ$ fulfill the requirements of Proposition 4.1. Both subgroups are by definition closed and connected. By Lemma 3.8 and Lemma 3.6 both $U_x$ and $T \times \Gamma$ coincide with their centralizers. Thus, $\varphi(U_x)$ and $\varphi(T \times \Gamma)$ are closed in $\mathrm{Aut}(X)$ by Corollary 2.8. This implies that

$$\overline{\varphi(U_{x,1})}^\circ \subseteq \varphi(U_x) \text{ and } \overline{\varphi(T)}^\circ \subseteq \varphi(T \times \Gamma).$$

Moreover, $\overline{\varphi(U_{x,1})}^\circ$ and $\overline{\varphi(T)}^\circ$ are commutative as $\varphi(U_x)$ and $\varphi(T \times \Gamma)$ are commutative. Set $t = (\sqrt{2}x, (\sqrt{2})^{-1}y, z) \in T$ and by Lemma 3.17 and we obtain

$$\varphi(U_{x,1}) = \{u \in \varphi(U_x) \mid \varphi(t^{-1}) \circ u \circ \varphi(t) = u^2\}$$

which shows that $\varphi(U_{x,1})$ is closed in $\varphi(U_x)$. Since $\varphi(U_x)$ is closed in $\mathrm{Aut}(X)$, the set $\varphi(U_{x,1})$ is closed in $\mathrm{Aut}(X)$ as well. By Lemma 1.3 $\varphi(U_{x,1})^\circ$ is closed in $\mathrm{Aut}(X)$.

Next, we check the properties (1) to (3) of Proposition 4.1. All non-trivial elements in $\varphi(U_{x,1})$ have an infinite order, because $U_{x,1}$ is isomorphic to $\mathbb{C}^+$ which has this property. Let $u = (x, x^{-1}p(z+gx)), z+gx) \in U_{x,1}, t = (cx, c^{-1}y, z) \in T$ and $\gamma = (x, \mu y, az+b) \in \Gamma$. By the proof of Lemma 3.8 and by Remark 3.5 it follows

$$\gamma u \gamma^{-1} = (x, x^{-1}p(z+agx), z+agx) \text{ and } tut^{-1} = (x, x^{-1}p(z+c^{-1}gx)), z+c^{-1}gx).$$

Thus, $T$ and $\Gamma$ normalize $U_{x,1}$. Furthermore, $\{g \in \overline{\varphi(T)}^\circ \mid g \circ h \circ g^{-1} = h\}$ is trivial for all $h \in \varphi(U_{x,1}) \setminus \{\mathrm{id}\}$. Hence, the third property of Proposition 4.1 is fulfilled. Recall that $\Gamma$ and $T$ act on $U_{x,1}$ by conjugation. Thus, $\varphi(T \times \Gamma)$ acts on $\varphi(U_{x,1})$ and in particular $\overline{\varphi(T)}^\circ \subseteq \varphi(T \times \Gamma)$ acts on $\varphi(U_{x,1})$. For $t \in \varphi(T \times \Gamma)$ we consider the morphism

$$\rho_t : \mathrm{Aut}(X) \to \mathrm{Aut}(X), x \mapsto txt^{-1}$$

which sends the identity to the identity. By Lemma 1.4

$$t\varphi(U_{x,1})^\circ t^{-1} \subseteq \varphi(U_{x,1})^\circ. \tag{5}$$

Moreover, $\overline{\varphi(T)}^\circ$ acts on $\varphi(U_{x,1})^\circ$ and we have the second property. Thus, we apply Proposition 4.1 and obtain

$$\overline{\varphi(T)}^\circ \cong \bigcup_{k=1}^{\infty} (\mathbb{C}^*)^{s_k} \text{ and } \varphi(U_{x,1})^\circ \cong \bigcup_{k=1}^{\infty} (\mathbb{C}^+)^{r_k}. \tag{6}$$

There exists $p \in \mathbb{N}$, such that there is no subgroup of $T \times \Gamma$ isomorphic to $(\mathbb{Z}/p\mathbb{Z})^2$. Thus, $\overline{\varphi(T)}^\circ$ cannot contain a subgroup isomorphic to $(\mathbb{Z}/p\mathbb{Z})^2$ either. This means that $s_k$ cannot be bigger than 1, because $(\mathbb{C}^*)^2$ contains a copy of $(\mathbb{Z}/n\mathbb{Z})^2$ for each natural $n$. Therefore, $\overline{\varphi(T)}^\circ$ is isomorphic to $\mathbb{C}^*$.

Now we claim that $\varphi(U_{x,1}) \subseteq \mathrm{Aut}(X)$ is isomorphic to $\mathbb{C}^+$. Indeed, by Example 1.36 all elements of $T$ are divisible. Further, since $\Gamma$ is a finite group, it does not contain any non-trivial divisible element. Since $\overline{\varphi(T)}^\circ \subseteq \varphi(T) \times \varphi(\Gamma)$ and since all elements of $\overline{\varphi(T)}^\circ \cong \mathbb{C}^*$ are divisible,

$$\overline{\varphi(T)}^\circ \subseteq \varphi(T). \tag{7}$$

Furthermore, $\varphi(T)$ acts transitively on $\varphi(U_{x,1}) \setminus \{\mathrm{id}\}$ because $U_{x,1}$ is a root subgroup with respect to $T$ of weight one. Hence, $\overline{\varphi(T)} \subseteq \varphi(T) \times \varphi(\Gamma)$ also acts transitively on $\varphi(U_{x,1}) \setminus \{\mathrm{id}\}$. By Lemma 2.11 there exists a countable set $C \subseteq \varphi(T)$ such that $\bigcup_{g \in C} \overline{\varphi(T)}^\circ g = \overline{\varphi(T)}$. Hence, $\overline{\varphi(T)}^\circ$ acts with a dense orbit on $\varphi(U_{x,1})^\circ$. As $\overline{\varphi(T)}^\circ \cong \mathbb{C}^*$ is one-dimensional, $\varphi(U_{x,1})$ is one-dimensional too and by (6) $\varphi(U_{x,1})^\circ \cong \mathbb{C}^+$.

By (5) $\varphi(t)\varphi(U_{x,1})^\circ \varphi(t)^{-1} \subseteq \varphi(U_{x,1})^\circ$ for any $\varphi(t) \in \varphi(T)$. Since $\varphi(T)$ acts transitively on $\varphi(U_{x,1}) \setminus \{\mathrm{id}\}$, we have

$$\varphi(U_{x,1}) \setminus \{\mathrm{id}\} = \{\varphi(t) \cdot x \cdot \varphi(t)^{-1} \mid t \in T\} \subseteq \varphi(U_{x,1})^\circ \setminus \{\mathrm{id}\},$$

where $x \in \varphi(U_{x,1})^\circ \setminus \{\mathrm{id}\}$. We conclude $\varphi(U_{x,1})^\circ \setminus \{\mathrm{id}\} = \varphi(U_{x,1}) \setminus \{\mathrm{id}\}$. This in particular means that $\mathbb{C}^+ \cong \varphi(U_{x,1})^\circ = \varphi(U_{x,1})$. Hence, $\varphi(U_{x,1})$ is a root subgroup of

$\mathrm{Aut}(X)$ with respect to $\overline{\varphi(T)}^\circ \subseteq \mathrm{Aut}(X)$. To finish the proof we need to show that $\overline{\varphi(T)}^\circ = \varphi(T)$. Indeed, by Corollary 3.15 $\overline{\varphi(T)}^\circ$ acts transitively on $\varphi(U_{x,1}) \setminus \{\mathrm{id}\}$ and since by (7) we have the inclusion $\overline{\varphi(T)}^\circ \subseteq \varphi(T)$, $\overline{\varphi(T)}^\circ$ acts on $\varphi(U_{x,1})$ with the trivial kernel. This finally implies $\overline{\varphi(T)}^\circ = \varphi(T)$ by Lemma 1.40. $\square$

Using Lemma 4.2 we prove that the dimension of $X$ cannot be smaller than 2.

**4.3 Lemma.** The dimension of $X$ is at least 2.

*Proof.* Assume towards a contradiction that $X$ is of dimension one. Note that by Lemma 4.2 $\varphi(U_{x,1}) \subseteq \mathrm{Aut}(X)$ is an algebraic subgroup isomorphic to $\mathbb{C}^+$. Let $x \in X$ be a point that is not a fixed $\varphi(U_{x,1})$-point. The stabilizer $\mathrm{Stab}_{\varphi(U_{x,1})}(x)$ is a closed non-trivial subgroup of $\varphi(U_{x,1})$. Moreover, Theorem 1.44 states that the orbit $\varphi(U_{x,1}) \cdot x$ of $x$ is a closed subset of $X$. Lemma 1.25 shows that a closed subset of a one-dimensional $X$ is finite or $X$.

By Lemma 1.39 $\varphi(U_{x,1}) \cdot x$ and $\varphi(U_{x,1}) / \mathrm{Stab}_{\varphi(U_{x,1})}(x)$ are isomorphic as varieties. Since the stabilizer $\mathrm{Stab}_{\varphi(U_{x,1}}(x) \subseteq \varphi(U_{x,1})$ is a closed subgroups by Lemma 2.7, it is either finite or the whole group by Lemma 1.25. Thus, the stabilizer is finite, because the stabilizer is not the whole group $\varphi(U_{x,1})$. This implies that the stabilizer is trivial, since $\varphi(U_{x,1})$ does not contain finite subgroups. In particular, this means that $\varphi(U_{x,1}) \cdot x = X$. Thus, $X$ has to be isomorphic to $\varphi(U_{x,1}) \cong \mathbb{C}^+$ as a variety. By Theorem 2.12 $D_p \cong \mathbb{A}^1$ which is a contradiction since $D_p$ is a surface. Thus, the dimension of $X$ cannot be 1. $\square$

Now we must show that the dimension of $X$ cannot be greater than 2. Before proving that, further notions are necessary.

**4.4 Definition.** Let $f$ be an element from $\mathcal{O}(X)$ and $G$ a subgroup of $\mathrm{Aut}(X)$. The function $f$ is called *$G$-invariant*, if $f(gx) = f(x)$ for all $g \in G$ and $x \in X$. The set of all $G$-invariants is denoted by $\mathcal{O}(X)^G$ and called the *invariant ring*.

A similar concept to invariants are the so-called *semi-invariants*. The function $f$ is called semi-invariant, if for all $g \in G$ and $x \in X$, there exists a character $\chi : G \to \mathbb{C}^*$, such that $f(gx) = \chi(g)f(x)$.

A vector subspace $V \subseteq \mathcal{O}(X)$ that is stable under $G$-action is called *multiplicity-free* with respect to $G$ if all $G$-semi-invariants from $V$ of the same weight are multiples of each other.

**4.5 Remark.** Let $f$ and $f'$ be two elements of $\mathcal{O}(X)^G$. Then $f + f'$ is again an invariant, since

$$(f + f')(gx) = f(gx) + f'(gx) = gf(x) + gf'(x) = g(f(x) + f'(x)) = g(f + f')(x)$$

for all $g \in G$ and $x \in X$. Analogously, we have

$$(f \cdot f')(gx) = f(gx) \cdot f'(gx) = gf(x) \cdot gf'(x) = g(f(x) \cdot f'(x)) = g(f \cdot f')(x)$$

for all $g \in G$ and $x \in X$. Thus, it is justified to call $\mathcal{O}(X)^G$ a ring.

**4.6 Lemma.** Let $X$ be an irreducible affine variety and let $T \subseteq \mathrm{Aut}(X)$ be a torus. Assume that there exists a root subgroup $U \subseteq \mathrm{Aut}(X)$ with respect to $T$, such that $\mathcal{O}(X)^U$ is multiplicity-free. Then $\dim T \leq \dim X \leq \dim T + 1$.

*Proof.* This lemma is proven in [7, Lemma 5.2]. □

We use Lemma 4.6 to prove Theorem 4.10.

**4.7 Definition.** Let $\lambda : \mathbb{C}^+ \to \mathrm{Aut}(X)$ be a non-trivial action. For an invariant $f \in \mathcal{O}(X)$ we define the *modification* $f \cdot \lambda$ as

$$(f \cdot \lambda)(s)x := \lambda(f(x)s)x \text{ for } s \in \mathbb{C} \text{ and } x \in X.$$

Note that $f \cdot \lambda$ again defines a $\mathbb{C}^+$-action on $X$.

If $U \subseteq \mathrm{Aut}(X)$ is a subgroup isomorphic to $\mathbb{C}^+$ and if $f \in \mathcal{O}(X)^U$ is an $U$-invariant, then we define the *modification* $f \cdot U$ of $U$ in a similar way. Choose an isomorphism $\lambda : \mathbb{C}^+ \to U$ and set $f \cdot U := (f \cdot \lambda)(\mathbb{C}^+)$, the image of the modified action.

**4.8 Remark.** Let $f \in \mathcal{O}(X)$ be a semi-invariant of weight $k$ with respect to a torus $T \subseteq \mathrm{Aut}(X)$ isomorphic to $\mathbb{C}^*$ and $U \subseteq \mathrm{Aut}(X)$ be a root subgroup with respect to $T$ of weight $l$. Then, the weight of $f \cdot U$ is the sum of the weight $k + l$.

**4.9 Lemma.** Let $f, g \in \mathcal{O}(X)$ be two semi-invariants with respect to a torus $T \subseteq \mathrm{Aut}(X)$ and $U \subseteq \mathrm{Aut}(X)$ be a root subgroups with respect to $T$. If the intersection $f \cdot U \cap g \cdot U$ is non-trivial, $f$ and $g$ are multiples of each other.

*Proof.* By Theorem 1.44 $U$-orbits are closed subvarieties of $X$. Since $U \cong \mathbb{C}^+$ is irreducible, $U$-orbits are also irreducible. Hence, a $U$-orbit is either a point or a one-dimensional closed subvariety of $X$. Thus, $U$ acts without fixed points on $X \setminus X^U$ and since $U$ is a one-dimensional algebraic group, i.e. $U$ does not contain a proper closed non-trivial subgroup, $U$ acts on $X \setminus X^U$ freely. Denote the set of fixed $U$-points in $X$ by $X^U$.

Assume that $f \cdot U \cap g \cdot U$ is not trivial, then there is $c \in \mathbb{C}$ that satisfies

$$\lambda(f(x)s)x = \lambda(g(x)cs)x \text{ for } s \in \mathbb{C}, x \in X.$$

Since we have a free action $f(x)s = g(x)cs$ for all $x \in X \setminus X^U$. Moreover, $X \setminus X^U \subseteq X$ is dense because $X \setminus X^U$ is an open subset in the irreducible set $X$. Thus, $f(x) = cg(x)$ for any $x \in X$. The proof follows. □

**4.10 Theorem.** The dimension of $X$ is at most 2.

*Proof.* By Lemma 4.2 $\varphi(U_{x,1})$ is a root subgroup with respect to $\varphi(T)$. To prove the theorem we need to show that $\mathcal{O}(X)^{\varphi(U_{x,1})}$ is multiplicity-free with respect to $\varphi(T)$.

Assume towards a contradiction that there are $g, f \in \mathcal{O}(X)^{\varphi(U_{x,1})}$ with the same weight that are not multiples of each other. We now consider subgroups $f \cdot \varphi(U_{x,1})$

and $g \cdot \varphi(U_{x,1})$ of $\mathrm{Aut}(X)$. Remark 4.8 shows that both have the same weight and by Lemma 4.9 they have a trivial intersection.

The next step is to consider the preimages $\varphi^{-1}(f \cdot \varphi(U_{x,1}))$, $\varphi^{-1}(g \cdot \varphi(U_{x,1})) \subseteq \mathrm{Aut}(D_p)$ and $\varphi^{-1}(\varphi(T)) = T$. Since $\varphi(T)$ acts transitively on $\varphi(U_{x,1}) \setminus \{\mathrm{id}\}$ and $f$ is an $\varphi(T)$-semi-invariant, $\varphi(T)$ acts transitively on $(f \cdot \varphi(U_{x,1})) \setminus \{\mathrm{id}\}$. This implies that $T$ acts transitively on $\varphi^{-1}(f \cdot \varphi(U_{x,1})) \setminus \{\mathrm{id}\}$. By Lemma 1.42 we obtain that $\varphi^{-1}(f \cdot \varphi(U_{x,1}))$ is a quasi-affine curve. Thus, $\varphi^{-1}(f \cdot \varphi(U_{x,1}))$ is a constructible set which is a group. By Lemma 1.41 (1) $\varphi^{-1}(f \cdot \varphi(U_{x,1})) = \overline{\varphi^{-1}(f \cdot \varphi(U_{x,1}))}$. Hence, $\varphi^{-1}(f \cdot \varphi(U_{x,1})) \subseteq \mathrm{Aut}(D_p)$ is an algebraic subgroup of dimension 1. To conclude that $\varphi^{-1}(f \cdot \varphi(U_{x,1}))$ is connected, we note that $\varphi^{-1}(f \cdot \varphi(U_{x,1})) \setminus \{\mathrm{id}\}$ is irreducible and

$$\big((\varphi^{-1}(f \cdot \varphi(U_{x,1})) \setminus \{\mathrm{id}\}\big) \cdot \big(\varphi^{-1}(f \cdot \varphi(U_{x,1})) \setminus \{\mathrm{id}\}\big) = \varphi^{-1}(f \cdot \varphi(U_{x,1})).$$

Thus, $\varphi^{-1}(f \cdot \varphi(U_{x,1}))$ can be written as the product of two irreducible quasi-affine subsets of $\mathrm{Aut}(D_p)$ and is in particular irreducible. By Lemma 1.32 $\varphi^{-1}(f \cdot \varphi(U_{x,1}))$ is either isomorphic to $\mathbb{C}^+$ or $\mathbb{C}^*$. Since $\varphi^{-1}(f \cdot \varphi(U_{x,1}))$ has no elements of finite order, it is isomorphic to $\mathbb{C}^+$. Therefore, $\varphi^{-1}(f \cdot \varphi(U_{x,1}))$ is a root subgroup of $\mathrm{Aut}(D_p)$ with respect to $T$. Analogously we obtain that $\varphi^{-1}(g \cdot \varphi(U_{x,1}))$ is a root subgroup of $\mathrm{Aut}(D_p)$ with respect to $T$.

By construction $\varphi^{-1}(f \cdot \varphi(U_{x,1}))$ and $\varphi^{-1}(g \cdot \varphi(U_{x,1}))$ are different root subgroups in $\mathrm{Aut}(D_p)$. Moreover, $T$ acts on $\varphi^{-1}(f \cdot \varphi(U_{x,1}))$ and $\varphi^{-1}(g \cdot \varphi(U_{x,1}))$ non-trivially. This implies that the kernels of such $T$-actions are finite as follows from Lemma 3.13. Since $\varphi : \mathrm{Aut}(D_p) \to \mathrm{Aut}(X)$ is the isomorphism, the cardinalities of the kernels of $T$-actions on $\varphi^{-1}(f \cdot \varphi(U_{x,1}))$ and $\varphi^{-1}(g \cdot \varphi(U_{x,1}))$ are the same as cardinalities of the kernels of $\varphi(T)$-actions on $f \cdot \varphi(U_{x,1})$ and $g \cdot \varphi(U_{x,1})$. Thus, $\varphi^{-1}(f \cdot \varphi(U_{x,1}))$ and $\varphi^{-1}(g \cdot \varphi(U_{x,1}))$ have the same weights up to a sign, because the kernel uniquely determines the weight up to a sign. Furthermore, $f \cdot \varphi(U_{x,1})$ and $g \cdot \varphi(U_{x,1})$ commute and thus $\varphi^{-1}(f \cdot \varphi(U_{x,1}))$ and $\varphi^{-1}(g \cdot \varphi(U_{x,1}))$ commute as well. Non-trivial elements from $U_x$ and $U_y$ do not commute by Lemma 3.8 which means that $\varphi^{-1}(f \cdot \varphi(U_{x,1}))$ and $\varphi^{-1}(g \cdot \varphi(U_{x,1}))$ are either both subgroups of $U_x$ or both subgroups of $U_y$. Hence, both weights have the same sign by Lemma 3.16 and $\varphi^{-1}(f \cdot \varphi(U_{x,1}))$ and $\varphi^{-1}(g \cdot \varphi(U_{x,1}))$ have the same weight.

This is a contradiction to Lemma 3.19, because root subgroups are uniquely determined by their weights in $\mathrm{Aut}(D_p)$. Hence, $f$ and $g$ are multiples of each other and $\mathcal{O}(X)^{\varphi(U_{x,1})}$ is multiplicity-free. Thus, the proof follows by Lemma 4.6 and the fact that the dimension of $\varphi(T) \cong \mathbb{C}^*$ is one. □

The next result is going to be used in the proof of Main Theorem A.

**4.11 Theorem.** Let $S$ be a normal affine surface and $D_p$ be a Danielewski surface for some $p \in \mathbb{C}[z]$. If $\mathrm{Aut}(S)$ and $\mathrm{Aut}(D_p)$ are isomorphic as groups, then $S$ is isomorphic to $D_q$ for some polynomial $q \in \mathbb{C}[z]$.

*Proof.* This theorem is proved in [11, Theorem 1]. □

*Proof of Main Theorem A.* Let $\deg p = 1$. Then, Lemma 2.13 states $D_p \cong \mathbb{A}^2$ and Theorem 2.12 proves our claim.

Hence, we assume $\deg p \geq 3$. We apply Lemma 4.3 and Theorem 4.10 to conclude $\dim X = 2$. Thus, $X$ is a surface. By Theorem 4.11, the proof follows. □

Let $p, q \in \mathbb{C}[z]$ be polynomials with only simple roots and degree at least 3. Formula (2) in Chapter 3 showed,

$$\mathrm{Aut}^\circ(D_p) = (U_x * U_y) \rtimes T.$$

By Remark 3.5, we observe that the action of $T$ on $U_x$ and $U_y$ is independent from the choice of the polynomial $p$. This implies that $\mathrm{Aut}^\circ(D_p)$ and $\mathrm{Aut}^\circ(D_q)$ are isomorphic as abstract groups. If $p$ and $q$ are generic, i.e. there is no automorphism of $\mathbb{C} = \mathbb{A}^1$ that permute their roots, then $\Gamma$ is trivial and we have

$$\mathrm{Aut}(D_p) \cong (x\mathbb{C}[x]^+ * y\mathbb{C}[y]^+) \rtimes (\mathbb{C}^* \rtimes \mathbb{Z}/2\mathbb{Z}) \cong \mathrm{Aut}(D_q).$$

Thus, Theorem A cannot be improved without additional assumptions. However, if we require the groups $\mathrm{Aut}(D_p)$ and $\mathrm{Aut}(D_q)$ to be isomorphic as ind-groups, we have the following result.

**4.12 Theorem.** Let $\varphi : \mathrm{Aut}^\circ(D_p) \to \mathrm{Aut}^\circ(D_q)$ be an isomorphism of ind-groups, where $p, q \in \mathbb{C}[z]$ are polynomials with simple roots. Then, the varieties $D_p$ and $D_q$ are isomorphic.

*Proof.* This theorem is proven in [9, Theorem 3]. □

*Proof of Main Theorem B.* Assume first $\deg p = 2$. Then $D_p$ is isomorphic to $\mathrm{SL}_2 / T$ by Lemma 2.13 and the claim follows by Proposition 2.14.

Consider now the case of $\deg p \neq 2$. By Remark 1.28 $X$ is normal. Hence, we apply Main Theorem A and we note that $X$ is isomorphic to $D_q$ for $q \in \mathbb{C}[z]$. By Lemma 1.27, we derive that $X \cong D_q$ has simple roots, since $X \cong D_q$ is smooth.

Thus, there exists an ind-group isomorphism from $\mathrm{Aut}(D_p)$ to $\mathrm{Aut}(D_q)$. By Lemma 2.10 $\varphi$ induces an ind-group isomorphism from $\mathrm{Aut}^\circ(D_p)$ to $\mathrm{Aut}^\circ(D_q)$. It follows from Theorem 4.12 that $D_p$ and $D_q$ are isomorphic as varieties. Thus, we conclude that $X$ is isomorphic to $D_p$ as a variety. □

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# Selbstständigkeitserklärung

Ich erkläre, dass ich die vorliegende Arbeit selbstständig und nur unter Verwendung der angegebenen Quellen und Hilfsmittel angefertigt habe.

Seitens des Verfassers bestehen keine Einwände, die vorliegende Masterarbeit für die öffentliche Benutzung im Universitätsarchiv zur Verfügung zu stellen.

Jena, den 18. Juli 2022