Automorphism groups without non-algebraic elements

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Preliminaries

Ind-group

An (affine) *ind-variety* is an inductive limit $V = \varinjlim V_i$ of closed embeddings $V_i \hookrightarrow V_{i+1}$ of (affine) algebraic varieties V_i, $i \in \mathbb{N}$. A *morphism* between ind-varieties $V = \varinjlim V_k$ and $W = \varinjlim W_m$ is a map $\phi : V \to W$ such that $\forall k > 0\ \exists m \in \mathbb{N}$ s.t. ϕ induces a morphism $V_k \to W_m$. An *ind-group* is an ind-variety endowed with a group structure such that maps of multiplication and taking inverse element are morphisms.

Automorphisms

Let X be an affine variety over an algebraically closed field $\mathbb{K}$ of characteristic zero. By Furter–Kraft, $\operatorname{Aut}(X)$ has a natural structure of an affine ind-group. An element $g \in \operatorname{Aut}(X)$ is called *algebraic* if there is an algebraic subgroup $G \subset \operatorname{Aut}(X)$ such that $g \in G$.

Derivations

A derivation $\partial \in \operatorname{Der} \mathcal{O}_X(X)$ is called

- *locally finite* if $\mathcal{O}_X(X)$ is a rational module under ∂.
- *semisimple* if ∂ acts diagonally w.r.t. some basis of $\mathcal{O}_X(X)$.
- *locally nilpotent* if $\forall f \in \mathcal{O}_X(X); \exists n \in \mathbb{N}$ s.t. $\partial^n(f) = 0$.

Lie algebra

Let $\operatorname{Aut}(X) = \varinjlim X_i$, then the tangent space $\mathrm{T}_e \operatorname{Aut}(X) = \bigcup_i \mathrm{T}_e X_i$ has a natural structure of a *Lie algebra* denoted $\operatorname{Lie}\operatorname{Aut}(X)$, since every $a \in \mathrm{T}_e \operatorname{Aut}(X)$ defines a unique left-invariant vector field on $\operatorname{Aut}(X)$. Moreover, there is an embedding into vector fields on X: $\operatorname{Lie}\operatorname{Aut}(X) \hookrightarrow \operatorname{Vec}(X) \cong \operatorname{Der} \mathcal{O}_X(X)$.

Structure theorem

The following conditions are equivalent:

1. all elements of $\operatorname{Aut}^\circ(X)$ are algebraic,
2. all elements of $\operatorname{Lie}\operatorname{Aut}(X)$ are locally finite,
3. $\operatorname{Aut}^\circ(X)$ is a nested ind-group, i.e., an inductive limit of algebraic groups,
4. $\operatorname{Aut}^\circ(X) = \mathbb{T} \ltimes \mathcal{U}$, where $\mathbb{T}$ is a maximal subtorus of $\operatorname{Aut}(X)$ and $\mathcal{U} \subset \operatorname{Aut}(X)$ is either an infinite-dimensional abelian unipotent closed ind-subgroup or trivial.

In particular, these conditions imply that all $\mathbb{G}_a$-actions on X are equivalent, if present.

Example

Let $X = \mathbb{A}^1_* \times \mathbb{A}^1 = \{(x, y) \mid x \in \mathbb{K}^*, y \in \mathbb{K}\}$, then

$$\operatorname{Aut}(X) = \{(x, y) \mapsto (t_1 x, t_2 y + P(x) \mid t_1, t_2 \in \mathbb{K}^*, P \in \mathbb{K}[x])\} \cong \mathbb{G}_m^2 \ltimes \mathcal{U}.$$

It satisfies all conditions in theorem above.

Proof idea

Given a pair of non-equivalent $\mathbb{G}_a$-actions, we present a non–locally finite element of $\operatorname{Lie}\operatorname{Aut}(X)$ and a non-algebraic element of $\operatorname{Aut}^\circ(X)$.
Let ∂_1, ∂_2 be locally nilpotent derivations corresponding to given $\mathbb{G}_a$-actions and $p \in X$ be a smooth point such that tangent vectors to $\mathbb{G}_a$-orbits at p are linearly independent. Then we may consider local coordinates $(x_1, \ldots, x_n)$ at p such that the least homogeneous components in formal power series are $\operatorname{LHC}(\partial_i) = \frac{\partial}{\partial x_i}$. Provided an additional open condition on p, we can find $f_i \in \ker \partial_i$ such that $\operatorname{LHC}(f_i \partial_i) = x_{3-i}^2 \frac{\partial}{\partial x_i}$, $i = 1, 2$. Then $f_1\partial_1 + f_2\partial_2 \in \operatorname{Lie}\operatorname{Aut}(X)$ is not locally finite, since $\operatorname{ord}[(f_1\partial_1 + f_2\partial_2)^n(x_1 + x_2)] = n + 1$. Similarly, the element $g = \exp(f_1\partial_1) \circ \exp(f_2\partial_2)$ acts on $\mathcal{O}_p(X)$ and turns out to be non–locally finite on it.